Birzeit University **Mathematics Department**

MATH 330 – Homework No. 3
Second Semester 2013/2014
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Name (بالعربية): ____Key______ **Student No.:** ___________ **Section No.:** ___

Question 1:
Find the quadratic spline $S(x)$ that interpolates the following data

x_i	0	1	2
y_i	0	3	3

solution

$$S(x) = \begin{cases} a_0 + a_1 x + a_2 x^2, & 0 \le x \le 1 \\ b_0 + b_1(x-1) + b_2(x-1)^2, & 1 \le x \le 2 \end{cases}$$

$S(0) = a_0 = 0 \Rightarrow \boxed{a_0 = 0}$

$S(1) = a_0 + a_1 + a_2 = b_0 \Rightarrow a_1 + a_2 = b_0$

$S(1) = b_0 = 3 \Rightarrow \boxed{b_0 = 3}$ and $\boxed{a_1 + a_2 = 3}$ Eq1

$S(2) = b_0 + b_1 + b_2 = 3 \Rightarrow \boxed{b_1 + b_2 = 0}$ Eq2

$$S'(x) = \begin{cases} a_1 + 2a_2 x, & 0 \le x \le 1 \\ b_1 + 2b_2(x-1), & 1 \le x \le 2 \end{cases}$$

$S'(1) = a_1 + 2a_2 = b_1 \Rightarrow \boxed{a_1 + 2a_2 = b_1}$ Eq3

$$S''(x) = \begin{cases} 2a_2, & 0 \le x \le 1 \\ 2b_2, & 1 \le x \le 2 \end{cases} \Rightarrow S''(1) = 2a_2 = 2b_2 \Rightarrow \boxed{a_2 = b_2}$$ Eq4

Solving Eq1, Eq2, Eq3, Eq4 $\Rightarrow a_1 = 4.5,\ a_2 = -1.5$
$b_1 = 1.5,\ b_2 = -1.5$

$$\Rightarrow \boxed{S(x) = \begin{cases} 4.5x - 1.5x^2, & 0 \le x \le 1 \\ 3 + 1.5(x-1) - 1.5(x-1)^2, & 1 \le x \le 2 \end{cases}}$$

Question 2:

a) Derive the normal equations (**without using linearization**) for obtaining the least squares circle $x^2 + y^2 = r^2$ with center (0,0) and radius r.
Hint: Assume that $f(x_k, y_k) = z_k + e_k$ *where* $f(x, y) = x^2 + y^2 - r^2$ *and solve the LS problem* $\sum_{k=1}^{n} e_k^2 = min!$

b) Using part (a) find the least squares circle for the following data

x_i	-1	-1	1	1
y_i	1	-3	-1	3

c) Compute the **rms** error $E_2(f)$ for part (b).
d) Compute the **average** error $E_1(f)$ for part (b).
e) Compute the **maximum** error $E_\infty(f)$ for part (b).

Solution

a) $\sum_{k=1}^{n} e_k^2 = min! \Rightarrow \sum_{k=1}^{n} (f(x_k, y_k) - z_k)^2 = min!$

$$\Rightarrow \underbrace{\sum_{k=1}^{n} (x_k^2 + y_k^2 - r^2 - z_k)^2}_{= E(r)} = min!$$

$$\frac{dE}{dr} = \sum_{k=1}^{n} 2(x_k^2 + y_k^2 - r^2 - z_k)(-2r) = 0$$

$$\Rightarrow \boxed{r^2 = \frac{\sum x_k^2 + \sum y_k^2 - \sum z_k}{n}}$$

b)

x_i	y_i	z_i	x_i^2	y_i^2	$f(x_i,y_i)$	e_i	e_i^2
-1	1	0	1	1	-4	-4	16
-1	-3	0	1	9	4	4	16
1	-1	0	1	1	-4	-4	16
1	3	0	1	9	4	4	16
—	—	0	4	20	—	—	64

$$r^2 = \frac{4 + 20 - 0}{4} = \frac{24}{4} = 6 \Rightarrow \boxed{x^2 + y^2 = 6}$$

c) $E_2(f) = \sqrt{\frac{1}{n} \sum_{k=1}^{n} e_k^2} = \sqrt{\frac{64}{4}} = 4 \Rightarrow \boxed{E_2(f) = 4}$

d) $E_1(f) = \frac{1}{n} \sum_{k=1}^{n} |e_k| = 4 \Rightarrow \boxed{E_1(f) = 4}$ e) $\boxed{E_\infty(f) = 4}$