Solutions of Homework 6: CS321, Fall 2010

Please show all steps in your work. Please be reminded that you should do your homework independently.

1. (10 points) Let $S_0(x) = c_0x + d_0$ be the linear polynomial defined on $[t_0, t_1]$, and $S_2(x) = c_2x + d_2$ the linear polynomial defined on $[t_2, t_3]$. We have

$$S_0'(x) = c_0, \qquad S_2'(x) = c_2,$$

and

$$S_0''(x) = 0, \qquad S_2''(x) = 0.$$

Assume the cubic spline polynomial defined on $[t_1, t_2]$ to be $S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$. It follows that

$$S_1'(x) = 3a_1x^2 + 2b_1x + c_1,$$

and

$$S_1''(x) = 6a_1x + 2b_1.$$

Since S, S' and S'' have to be continuous on $[t_0, t_3]$, we must have

$$S_0''(t_1) = S_1''(t_1), \qquad S_1''(t_2) = S_2''(t_2).$$

Hence

$$6a_1t_1 + 2b_1 = 0, \qquad 6a_1t_2 + 2b_1 = 0. \tag{1}$$

These two equations lead to

$$6a_1(t_1 - t_2) = 0.$$

Since $t_1 \neq t_2$, we have $a_1 = 0$. From Eq. (1), we have $b_1 = 0$.

We can now conclude that $S_1(x) = c_1x + d_1$, so S is also a linear polynomial on $[t_1, t_2]$.

2. (10 points) Find a quadratic spline interpolant for these data

x	-1	0	$1/2$	1	2	$5/2$
y	2	1	0	1	2	3

Solution. Suppose the quadratic spline interpolant has the following form

$$Q(x) = \begin{cases} Q_0(x), & x \in [-1, 0] \\ Q_1(x), & x \in [0, 1/2] \\ Q_2(x), & x \in [1/2, 1] \\ Q_3(x), & x \in [1, 2] \\ Q_4(x), & x \in [2, 5/2] \end{cases}$$

Define $z_i = Q_i'(t_i)$, the following is the formula for Q_i

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i.$$

It follows that

$$z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right), \qquad (0 \le i \le n-1).$$

By setting $z_0 = 0$, we can compute recursively,

$$\begin{aligned}
z_1 &= -0 + 2\left(\frac{1-2}{0-(-1)}\right) = -2,\\
z_2 &= -(-2) + 2\left(\frac{0-1}{1/2-0}\right) = -2,\\
z_3 &= -(-2) + 2\left(\frac{1-0}{1-1/2}\right) = 6,\\
z_4 &= -6 + 2\left(\frac{2-1}{2-1}\right) = -4,\\
z_5 &= -(-4) + 2\left(\frac{3-2}{5/2-2}\right) = 8.
\end{aligned}$$

Hence, the quadratic spline interpolant is

$$Q(x) = \begin{cases} Q_0(x) = -(x+1)^2 + 2, & x \in [-1, 0] \\ Q_1(x) = -2x + 1, & x \in [0, 1/2] \\ Q_2(x) = 8(x - 1/2)^2 - 2(x - 1/2), & x \in [1/2, 1] \\ Q_3(x) = -5(x-1)^2 + 6(x-1) + 1, & x \in [1, 2] \\ Q_4(x) = 12(x-2)^2 - 4(x-2) + 2, & x \in [2, 5/2] \end{cases}$$

3. (10 points) Determine if this function is a quadratic spline? Explain why or why not.

$$Q(x) = \begin{cases} x & -\infty < x \le 1 \\ x^2 & 1 \le x \le 2 \\ 4 & 2 \le x < \infty \end{cases}$$

Solution. $Q(x)$ is not a quadratic spline. The domain of definition is $(-\infty, \infty)$ which is not finite. Furthermore,

$$\begin{aligned}
\lim_{x \to 1^-} Q'(x) &= \lim_{x \to 1^-} 1 = 1,\\
\lim_{x \to 1^+} Q'(x) &= \lim_{x \to 1^+} 2x = 2.
\end{aligned}$$

It follows that $Q'(x)$ is not continuous at $x = 1$, which violates the definition of a quadratic spline.

4. (10 points) Determine the parameters a, b, c, d and e so that S is a natural cubic spline

$$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [0,1] \\ (x-1)^3 + ex^2 - 1 & x \in [1,2] \end{cases}$$

Solution. We first compute the derivatives of the individual functions

$$\begin{aligned} S_0'(x) &= b + 2c(x-1) + 3d(x-1)^2 \\ S_0''(x) &= 2c + 6d(x-1) \\ S_1'(x) &= 3(x-1)^2 + 2ex \\ S_1''(x) &= 6(x-1) + 2e \end{aligned}$$

We will make use of the continuity condition and the definition of the natural cubic spline.

From $S_0(1) = S_1(1)$, we have $a = c - 1$.

From $S_0'(1) = S_1'(1)$, we have $b = 2e$.

From $S_0''(1) = S_1''(1)$, we have $2c = 2e$, with $c = e$.

We also have

$$S_0''(0) = S''(2) = 0$$

$$2c - 6d = 0, \qquad 2e + 6 = 0,$$

we have $e = -3$, $c = -3$. Then $a = e - 1 = -4$, $b = 2e = -6$, and $d = c/3 = -1$.

5. (10 points) Determine the coefficients so that the function

$$S(x) = \begin{cases} x^2 + x^3 & 0 \le x \le 1 \\ a + bx + cx^2 + dx^3 & 1 \le x \le 2 \end{cases}$$

is a cubic spline and has the property $S_1'''(x) = 12$.

Solution. The derivatives are

$$\begin{aligned} S_0'(x) &= 2x + 3x^2 \\ S_0''(x) &= 2 + 6x \\ S_1'(x) &= b + 2cx + 3dx^2 \\ S_1''(x) &= 2c + 6dx \\ S_1'''(x) &= 6d \end{aligned}$$

Given the condition $S_1'''(x) = 12$, we have $d = 2$.

By continuity, we have $S_0'(1) = S_1'(1)$, which yields $5 = b + 2c + 3d$.

From $S_0''(1) = S_1''(1)$, we have $8 = 2c + 6d$.

We can solve the above two equations and get $c = -2$ and $b = 3$.

From $S_0(1) = S_1(1)$, we have $2 = a + b + c + d$, which gives us $a = -1$.