جامعة بيرزيت
BIRZEIT UNIVERSITY

Name: Key

Number:

MATH 330

First Semester

midterm Exam

2014/2015

1) (30points) Consider the equation $f(x) = x^2 - \sin x - 1 = 0$

(a) Find a function $g(x)$ that has a unique fixed point in $[0, \pi/2]$

Show both claims.

(b) Does the fixed point iterations for the above function g(x) converges why.

(c) Find the number of iterations needed to get accuracy 10^{-2} starting with $p_0 = 1.5$

(d) Estimate the fixed point starting with $p_0 = 1.5$ with accuracy 10^{-2}.

(a) $x = \sqrt{1+\sin x} \Rightarrow g(x) = \sqrt{1+\sin x}$ on $[0, \frac{\pi}{2}]$.

. $g(x)$ is continuous on $[0, \frac{\pi}{2}]$, $1+\sin x > 0$, $\forall x \in [0, \frac{\pi}{2}]$

. $g(x)$ is increasing function on $[0, \frac{\pi}{2}]$

$$\left.\begin{array}{l} g(0) = 1 \\ g(\frac{\pi}{2}) \simeq 1.41 \end{array}\right\} \Rightarrow g(x) \in [0, \tfrac{\pi}{2}], \ \forall x \in [0, \tfrac{\pi}{2}]$$

then there exists at least 1 fixed point in $[0, \frac{\pi}{2}]$

To prove Uniqueness: $|\cos x| \leq 1$, $\forall x \in [0, \frac{\pi}{2}]$

$$g'(x) = \frac{\cos x}{2\sqrt{1+\sin x}} \Rightarrow |g'(x)| \leq \left| \frac{1}{2\sqrt{1+\sin x}} \right| \leq \frac{1}{2}.$$

decreasing on $[0, \frac{\pi}{2}]$

$\Rightarrow |g'(x)| \leq \frac{1}{2} < 1$ in $[0, \frac{\pi}{2}]$.

So $g(x)$ has a unique fixed point in $[0, \frac{\pi}{2}]$

(b) Yes, since $|g'(x)| \leq k < 1, \forall x \in [0, \frac{\pi}{2}]$,

then by F.P.I theorem, the sequence $\{p_n\}$ converges to the unique fixed point.

(c) $p_0 = 1.5$, $|p - p_n| \leq 10^{-2}$

$$|p - p_n| \leq \frac{k^n |p_1 - p_0|}{1-k}$$

, $p_1 = g(p_0) \cong 1.413327$

$k = \frac{1}{2}$

then $\frac{(\frac{1}{2})^n |1.413327 - 1.5|}{1 - \frac{1}{2}} \leq 10^{-2}$

$\Rightarrow (\frac{1}{2})^n (0.0867) \leq 5 \times 10^{-3} \Rightarrow 17.34 \leq 2^n$

then $\ln 17.34 \leq n \ln 2 \Rightarrow 4.11 \leq n \Rightarrow \boxed{n = 5}$

(d)

k	p_k	$\lvert p_{k+1} - p_k \rvert$
0	1.5	-
1	1.413327	8.6×10^{-2}
2	1.40983	3.495×10^{-3}

$\longrightarrow$ $\overset{\text{error}}{\leq} 10^{-2}$

(Q#2) (20points) Consider the same equation $f(x) = x^2 - sinx - 1 = 0$

(1) Use Newton Method to estimate the solution of the above equation with accuracy 10^{-3}.

(2) Find the order of convergence of the above iteration both theoretically and numerically.

Newton Method:

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

(1)

n	P_n	$\lvert P_{n+1} - P_n \rvert$
0	1.5	—
1	1.4138	8.6×10^{-2}
2	1.40963	4.16×10^{-3}
3	1.409624	5×10^{-6}

$< 10^{-3}$

(2) Theoritically: Assume the root is ≈ 1.409624

$f'(1.\cancel{4}09624) = 2.65877 \neq 0 \Rightarrow M = 1$ simple root

& $R = 2$, $A = \left| \frac{f''(p)}{2f'(p)} \right| \approx 0.5617$.

Estimating the error:

k	$\frac{\lvert e_{k+1} \rvert}{\lvert e_k \rvert^2}$
1	0.51132 --
2	0.5617

(3) (18points) Consider the same equation $f(x) = x^2 - sinx - 1 = 0$

(a) Use false position method to find p_2, p_3 given $p_0 = 0, \quad p_1 = \frac{\pi}{2}$

(b) Use bisection method to estimate the root, find c_0, c_1 only, starting with $[0, \pi/2]$

(c)Use secant Method to find the 1st two iterations starting with $p_0 = 0, \quad p_1 = \pi/2$

(a) $C_n = b_n - \dfrac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$

$f(0) = -1$

$f(\frac{\pi}{2}) = 0.467$

$C_0 = b_0 - \dfrac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)} = \frac{\pi}{2} - \dfrac{f(\frac{\pi}{2})(\frac{\pi}{2} - 0)}{f(\frac{\pi}{2}) - f(0)} \approx 1.07046$

$f(c_0) = -0.7315 \Rightarrow a_1 = 1.07046, \; b_1 = \frac{\pi}{2}$

$[a_1, b_1] = [1.07046, \frac{\pi}{2}]$

$c_1 = \frac{\pi}{2} - \dfrac{f(\frac{\pi}{2})(\frac{\pi}{2} - 1.07046)}{f(\frac{\pi}{2}) - f(1.07046)} \approx 1.3757$

(b) $C_n = \dfrac{b_n + a_n}{2}$

$C_0 = \frac{\pi}{4} \approx 0.785398$

$c_1 = \frac{3\pi}{8} = 1.178097$

a_n		c_n		b_n		$f(c_n)$
0	−	$\frac{\pi}{4}$	−	$\frac{\pi}{2}$	+	−1.0
$\frac{\pi}{4}$	−	$\frac{3\pi}{8}$		$\frac{\pi}{2}$	+	

(c) The same as part (a)

(4) (20points)

(a) Solve the following system using L-U Factorization

$$\begin{pmatrix} 1 & 1 & 6 \\ -1 & 2 & 9 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 10 \end{pmatrix}$$

(b)Find the cost of (a)

(c)Find the cost of L-U for nxn matrix A

(a) $$\begin{pmatrix} 1 & 1 & 6 \\ -1 & 2 & 9 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{G.E} \begin{pmatrix} 1 & 1 & 6 \\ 0 & 3 & 15 \\ 0 & 3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 6 \\ 0 & 3 & 15 \\ 0 & 0 & 12 \end{pmatrix} =$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix},$$

Solve $Ux = y$ where $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 10 \end{bmatrix} \Rightarrow$$ solving this system:

$y_1 = 7,\ y_2 = 9,\ y_3 = 12$

Now solve $Ux = y$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 7 \\ 0 & 3 & 15 & 9 \\ 0 & 0 & 12 & 12 \end{array}\right] \Rightarrow$$

$x_3 = 1$

$x_2 = -2$

$x_1 = 3$

(b)

Step	$+,-$	$\times,\div$
1	4	6
2	1	2

$2+(2\times 2)$ cost $\sum +,- = 5$

$1+(1)(1)$ cost $\sum \times,\div = 8$

cost F.S $= n^2-n = 6$

cost B.S $= n^2 = 9$

total cost $= 5+8+9+6 = 28$

(C)

Step	$+,-$	$\times,\div$
1	$(n-1)^2$	$(n-1)^2+(n-1)$
2	$(n-2)^2$	$(n-2)^2+(n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k)^2$	$(n-k)^2+(n-k)$
$\vdots$	$\vdots$	$\vdots$
$(n-1)$	1	$1+1$

$(n^2-n)(2n-1)$

$\frac{n(n-1)(2n-1)}{6}$

$\frac{2n^3 - n^2 - 2n^2 + n}{6} + \frac{n(n+1)}{2}$

cost of $+,-$: $\sum_{k=1}^{n-1}(n-k)^2$, assume $n-k=p$

$$\sum_{p=1}^{n-1} p^2 = \frac{2n^3-3n^2+n}{6}$$

cost of $\times,\div = \sum_{p=1}^{n-1} p^2+p = \frac{n^3-n}{3}$

$\Rightarrow$ cost $+,-,\times,\div = \frac{4n^3-3n^2-n}{6}$

Cost of B.S $= n^2$

Cost of F.S $= n^2-n$

$\frac{2n^3-\cancel{3n^2}+n+\cancel{3n^2}+3}{6}$

$\frac{2n^3+n+3}{6}$

Total cost: $\frac{4n^3+9n^2-7n}{6}$

$O(n^3)$

(5) (12 points)Using Newton Method to estimate the solution of the following system starting with (1,1) . Find only one iteration

$7x^3 - 10x - y - 1 = 0$

$8y^3 - 11y + x - 1 = 0$

$$f_1(x,y) = 7x^3 - 10x - y - 1$$

$$f_2(x,y) = 8y^3 - 11y + x - 1$$

$$J = \left. \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \right|_{(1,1)} = \left. \begin{pmatrix} 21x^2 - 10 & -1 \\ 1 & 24y^2 - 11 \end{pmatrix} \right|_{(1,1)} = \begin{pmatrix} 11 & -1 \\ 1 & 13 \end{pmatrix}$$

Inverse method:

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - J^{-1}\Big|_{(p_0,q_0)} \begin{bmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{bmatrix}, \quad f_1(1,1) = -5, \quad f_2(1,1) = -3$$

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{13}{144} & \frac{1}{144} \\ \frac{-1}{144} & \frac{11}{144} \end{bmatrix} \begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -0.47222 \\ -0.19444 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.47222 \\ 0.19444 \end{bmatrix}$$

$$\Rightarrow (p_1, q_1) = (1.47222, 1.19444)$$