Birzeit University **Mathematics Department**

MATH 330 – Quiz No. 4
Second Semester 2013/2014
Prepared by: Dr. Hani Kabajah
Date: 15/5/2014

Name (بالعربية): ___KEY______ **Student No.:** ____________ **Section No.:** ___

Question

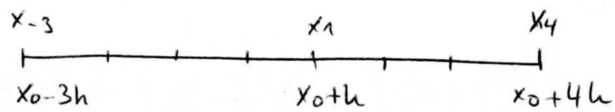
a) Derive an approximation for $f''(x_0)$ using the nodes $x_{-3} = x_0 - 3h$, $x_1 = x_0 + h$, $x_4 = x_0 + 4h$.
You may use the Lagrange or the Newton Polynomial.

b) Derive the truncation error $E_{trunc}(f, h)$ for part a).

c) Find the optimal step-size h_{opt} using parts a) and b).

Solution

a) Nodes



$$P(x) = a_0 + a_1(x - x_{-3}) + a_2(x - x_{-3})(x - x_1)$$

$$P'(x) = a_1 + a_2\left[(x - x_{-3}) + (x - x_1)\right]$$

$$P''(x) = a_2[1+1] = 2a_2 = 2\left(\frac{\dfrac{f(x_4) - f(x_1)}{x_4 - x_1} - \dfrac{f(x_1) - f(x_{-3})}{x_1 - x_{-3}}}{x_4 - x_{-3}}\right)$$

$$P''(x) = 2\,\frac{1}{7h}\left(\frac{f_4 - f_1}{3h} - \frac{f_1 - f_{-3}}{4h}\right) = \frac{2}{7h}\left(\frac{4f_4 - 7f_1 + 3f_{-3}}{12h}\right)$$

$$P''(x) = \frac{4f_4 - 7f_1 + 3f_{-3}}{42h^2} \Rightarrow \boxed{f''(x_0) \cong \frac{3f_{-3} - 7f_1 + 4f_4}{42h^2}}$$

b) $E_2(x) = (x-x_{-3})(x-x_1)(x-x_4)\dfrac{f^{(3)}(c)}{3!}$

$\{$Let $\quad t = x - x_1 \}$

$$E_2(t+x_1) = (t+x_1-x_{-3})(t)(t+x_1-x_4)\frac{f^{(3)}(c)}{6}$$

$$= t(t+4h)(t-3h)\frac{f^{(3)}(c)}{6}$$

$$= (t^2+4ht)(t-3h)\frac{f^{(3)}(c)}{6}$$

$$E_2'(t+x_1) = \left[(2t+4h)(t-3h) + (t^2+4ht)\right]\frac{f^{(3)}(c)}{6}$$

$$E_2''(t+x_1) = \left[(2t+4h) + 2(t-3h) + (2t+4h)\right]\frac{f^{(3)}(c)}{6}$$

$$E_2''(x_0) = E''(x_1-h) = \left[(-2h+4h)+2(-h-3h)+(-2h+4h)\right]\frac{f^{(3)}(c)}{6}$$

$$= (2h-8h+2h)\frac{f^{(3)}(c)}{6} = \frac{-4hf^{(3)}(c)}{6}$$

$$\Rightarrow \boxed{E(f,h) = -\frac{2hf^{(3)}(c)}{3}}$$

c) $B(h) = \dfrac{14\varepsilon}{42h^2} + \dfrac{2hM}{3} = \dfrac{\varepsilon}{3h^2} + \dfrac{2hM}{3}$

$$B'(h) = \frac{-2\varepsilon}{3h^3} + \frac{2M}{3} = 0 \Rightarrow \frac{2\varepsilon}{3h^3} = \frac{2M}{3}$$

$$\Rightarrow h^3 = \frac{6\varepsilon}{6M} \Rightarrow \boxed{h_{opt} = \left(\frac{\varepsilon}{M}\right)^{1/3}}$$