Q1) $f(x) = \frac{1}{x}$

$n+1 = 3 \Rightarrow n = 2$

0.1	10
0.2	5
0.3	3.33
$\uparrow$ x_k	$\uparrow$ y_k

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k}(x)$$

$$= y_0 L_{2,0} + y_1 L_{2,1} + y_2 L_{2,2}$$

$$\Rightarrow P_2 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\Rightarrow P_2(x) = 500(x-0.3)(x-0.2)$$

$$- 500(x-0.3)(x-0.1)$$

$$+ 166.5(x-0.2)(x-0.1)$$

Q2) Uniform Partition $\Rightarrow |E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}}$

$M_3 = \text{Max}|f'''(x)|$

$f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} \Rightarrow |f'''(x)| = \left|\frac{-6}{x^4}\right| = \frac{6}{x^4}$

Max at $0.1 = x$

$$\Rightarrow M_3 = \frac{6}{(0.1)^4} = 60{,}000$$

$$|E_2| \le \frac{(0.1)^3 \, 60000}{9\sqrt{3}} = 3.849$$

Q3) $f(x) = x^3 - 3x$ $\quad 4 = n+1$ points $\Rightarrow n = 3$

0	0
1	-2
2	2
3	18

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$+ a_3(x-x_0)(x-x_1)(x-x_2)$$

$$a_0 = y_0 = 0$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = -2$$

$$a_2 = \frac{\frac{y_1-y_0}{x_1-x_0} + \frac{y_2-y_1}{x_2-x_1}}{x_2 - x_0} = 3$$

$$a_3 = \frac{3 + \frac{y_2-y_1}{x_2-x_1} + \frac{y_3-y_2}{x_3-x_2}}{x_3 - x_0} = 1$$

$$\Rightarrow P_3(x) = -2(x-0) + 3(x)(x-1) + (x)(x-1)(x-2)$$

Q4) $P_3(x)$ and $f(x)$: - have the same values at nodes

i.e. $P_3(x_k) = f(x_k)$; $k = 0, 1, 2, 3$

$P_3(0) = f(0) = 6$

$P_3(1) = f(1) = -2$

$P_3(2) = f(2) = 2$

$P_3(3) = f(3) = 18$

- The both are polynomials of degree 3

Q5) $f(x) = Ax^3 + Bx$

$$E(A,B) = \sum_{i=1}^{n} (f(x_i) - y_i)^2$$

$$\frac{\partial E}{\partial A} = 2\sum (Ax_i^3 + Bx_i - y_i)(x_i^3) = 0$$

$$A\sum x_i^6 + B\sum x_i^4 = \sum x_i^3 y_i \quad \text{— (1)}$$

$$\frac{\partial E}{\partial B} = 2\sum (Ax_i^3 + Bx_i - y_i)(x_i) = 0$$

$$A\sum x_i^4 + B\sum x_i^2 = \sum x_i y_i \quad \text{— (2)}$$

Q6)

x_k	y_k
1	-2
2	2
3	18

$\sum x_i y_i = 56$ $\qquad \sum x_i^3 y_i = 500$

$\sum x_i^2 = 14$ $\qquad \sum x_i^4 = 98$

$\sum x_i^4 = 98$ $\qquad \sum x_i^6 = 794$

$794A + 98B = 500$

$98A + 14B = 56$

$$\begin{pmatrix} 794 & 98 \\ 98 & 14 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 500 \\ 56 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0.00925 & -0.0648 \\ -0.0648 & 0.525 \end{pmatrix}\begin{pmatrix} 500 \\ 56 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0.9962 \\ -3 \end{pmatrix} \approx \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Q10)

x_k	y_k
1	2
3	4
5	2

$L_{2,1}(x) = \frac{3}{4}$ Find x

$$L_{2,1}(x) = \frac{(5-x)(x-1)}{4} = \frac{3}{4}$$

$$(5-x)(x-1) = 3$$

$$-x^2 + 6x - 8 = 0$$

$$(x-2)(x-4) = 0$$

$$\boxed{x = 2, 4}$$

$x = 2$ or 4

Q7) $f(x) = x^3 - 3x$

x_k	y_k	$f(x_k)$	$\lvert e_k \rvert = \lvert f(x_k) - y_k \rvert$	$\lvert e_k \rvert^2$
0	1	0	1	1
1	-3	-2	1	1
2	5	2	3	9
3	15	18	3	9

ME $E_\infty = \max \lvert e_k \rvert = 3$

AVE $E_1 = \frac{1}{4} \sum_{k=1}^{4} \lvert e_k \rvert = \frac{8}{4} = 2$

RMSE $E_2 = \sqrt{\frac{1}{4} \sum_{k=1}^{4} \lvert e_k \rvert^2} = \sqrt{\frac{1}{4}(1+1+9+9)} = \sqrt{5}$

Q8) $f(x) = Cx + Dx^3 = y$

$y = Cx + Dx^3 \Rightarrow \frac{y}{x} = Dx^2 + C$

$y' = Dx' + C$

Q9) $\begin{array}{cc} -1 & 22 \\ 1 & 3 \end{array}$ $f'(1) = 0, \; f'(3) = 24$

$n+1 = 2 \Rightarrow n = 1$

$S_0(x) = A_0(x-1)^3 + B_0(x-1)^2 + C_0(x-1) + D_0 \quad , \; 1 \le x \le 3$

$S_0(1) = \boxed{D_0 = -1} \; , \; S_0(3) = 8A_0 + 4B_0 + 2C_0 - 1 = 22$

$0 = f'(1) = S_0'(1) \Rightarrow \boxed{C_0 = 0}$

$24 = f'(3) = S_0'(3) \Rightarrow 3A_0(\underset{3}{x}-1)^2 + 2B_0(\underset{3}{x}-1) + C_0 = 24$

$\Rightarrow 12A_0 + 4B_0 = 24$

$8A_0 + 4B_0 = 23 \quad —(1)$

$12A_0 + 4B_0 = 24 \quad —(2)$

$4A_0 = 1 \Rightarrow \boxed{A_0 = \frac{1}{4}}$

$2 + 4B_0 = 23$

$\boxed{B_0 = \frac{21}{4}}$