Birzeit University

Mathematics Department

Math 330 - Second Exam

Second Semester 2013/2014

StudentName:.. Key ....Number:.. ection:

Consider the function $f(x) = x^2 - \frac{2}{x}$ and the data $(1,-1), (2,3), (5,24.6)$, to answer the following questions:

$x_0 \quad x_1 \quad x_2$

Question 1. .

a) (5points) *Estimate* $f(4)$ *using Lagrange polynomial* $p_2(x)$.

b) (5points) *Estimate* $f(4)$ *using Newton polynomial* $p_2(x)$.

c) (8points) *Find an upper bound for* $E_2(x)$ *for all* $x \in [1,5]$.

Notice that the given intervals are of different lengths.

$\boxed{a}$

2 $$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

1 $$P_2(4) = \frac{(4-2)(4-5)}{(1-2)(1-5)}(-1) + \frac{(4-1)(4-5)}{(2-1)(2-5)}(3) + \frac{(4-1)(4-2)}{(5-1)(5-2)} 24.6$$

1 $$= 0.5 + 3 + 12.3 = 15.8$$

$\boxed{b}$

x_K	$f(x_K)$	1's	2nd
1	-1		
2	3	4	
5	24.6	7.2	0.8

$$P_1(x) = a_0 + a_1(x-x_0)$$
$$= -1 + 4(x-1)$$

2 $$P_2(x) = P_1(x) + a_2(x-x_0)(x-x_1)$$

2 $$= -1 + 4(x-1) + 0.8(x-1)(x-2)$$

1 $$P_2(4) = -1 + 12 + 0.8 * 6$$
$$= 15.8$$

$$|E_2(x)| \leq \frac{|(x-x_0)(x-x_1)(x-x_2)|}{3!} \max |f^{3}(x)|$$

2

let $\phi(x) = (x-x_0)(x-x_1)(x-x_2) = (x-1)(x-2)(x-5)$

$= (x-1)(x^2-7x+10)$

$= x^3-7x^2+10x-x^2+7x-10$

$= x^3-8x^2+17x-10$

3

$\phi'(x) = 3x^2-16x+17 = 0$

$x = \dfrac{16 \pm \sqrt{256-204}}{6} = \dfrac{16 \pm 7.2111}{6}$ $\quad x = 3.8685$ or $x = 1.464817$

maximum of $|\phi(x)|$

$= |\phi(3.8685)|$

$= |-6.0646| = 6.0646$

$+++ \; --- \; --- \; ++$

$1 \quad 1.46\ldots \quad 2 \quad 3.268\ldots \quad 5$

$f(x) = x^2 - \dfrac{2}{x}$

$f'(x) = 2x + \dfrac{2}{x^2}$

$f''(x) = 2 - \dfrac{4}{x^3}$

$f'''(x) = \dfrac{12}{x^4}$

maximum of $|f'''(x)|$

$= |f'''(1)| = 12$ 2

$|E_2(x)| \leq \dfrac{6.0646 * 12}{6} = 12.1292$ 1

$0.5 + 3 + 12.3$

$a_0 = 1, a_1 = 4, a_2 = 0.5$

Q2 : $\boxed{c}$

$s_0 = 0 \quad s_1 \quad s_2 = 0$

$x_0 \; g_0 \; x_1 \; g_1 \; x_2$

$2s_1(h_0+h_1) = 6\left[F[x_1,x_2] - F[x_0,x_1]\right]$

$8s_1 = 19.2 \Rightarrow \boxed{s_1 = 2.4}$ 2

$h_0 = x_1 - x_0 = 1$

$h_1 = x_2 - x_1 = 3$

$F[x_1,x_2] = 7.2$

$F[x_0,x_1] = 4$

طريقة حسابها فوق السؤال

$b_i = \dfrac{s_i}{2} \quad b_0 = 0 \quad b_1 = 1.2$

$a_i = \dfrac{s_{i+1}-s_i}{6h_i} \quad a_0 = \dfrac{2.4}{6*1} = 0.4$, $a_1 = \dfrac{-2.4}{6*3} = -0.1333$

$c_0 = 4 - \dfrac{0 + 2.4*1}{6} = 3.6$, $c_1 = 7.2 - \dfrac{2*3*2.4-0}{6}$ 2

$= 4.8$

$d_0 = -1 = f(x_0)$, $d_1 = f(x_1) = 3$.

$$g(x) = \begin{cases} g_0(x) = 0.4(x-1)^3 + 3.6(x-1) + -1 \\ g_1(x) = -0.1333(x-2)^3 + 1.2(x-2)^2 + 4.8(x-2) + 3 \end{cases}$$ 1

$g_1 \Rightarrow f(4) = -0.1333(8) + 1.2(4) + 4.8(2) + 3 = 16.336$

[redacted]

$$F[X_1, X_2] = \frac{f(X_2) - f(X_1)}{X_2 - X_1} = \frac{24.6 - 3}{3} = 7.2$$

$$F[X_0, X_1] = \frac{f(X_1) - f(X_0)}{X_1 - X_0} = \frac{3 - -1}{1} = 4$$

Question 2. *consider the following data* $(1, -1), (2, 3), (5, 24.6)$

a) (6points) *Using above data, Find the least-square fit of the equation* $y = Ax^2 + \frac{B}{x}$, *then use it to estimate* $f(4)$.

b) (6points) *Use Linearization to find the best fit of the given data, then use it to estimate* $f(4)$.

موجود قبل ⇒ c) (6points) *Find the natural cubic spline for the given data, then use it to estimate* $f(4)$.

$\boxed{a}$

$$E(A,B) = \sum_{K=1}^{n} \left(A X_K^2 + \frac{B}{X_K} - Y_K\right)^2$$

$$\frac{dE}{dA} = \sum_{K=1}^{n} 2\left(A X_K^2 + \frac{B}{X_K} - Y_K\right) X_K^2 = 0$$

$$A \sum_{K=1}^{n} X_K^4 + B \sum X_K = \sum Y_K X_K^2$$

$A\Sigma x^4 + B\Sigma x = \Sigma x^2 y$

$A\Sigma x + B\Sigma \frac{1}{x^2} = \Sigma \frac{y}{x}$

2

$$\frac{dE}{dB} = \sum_{K=1}^{n} 2\left(A X_K^2 + \frac{B}{X_K} - Y_K\right) \frac{1}{X_K} = 0$$

$x^4, \frac{1}{x^2}, x^2 y, y/x$

$$A \sum_{K=1}^{n} X_K + B \sum \frac{1}{X_K^2} = \sum \frac{Y_K}{X_K}$$

X_K	Y_K	X_K^4	$\frac{1}{X_K^2}$	$\frac{Y_K}{X_K}$	$Y_K X_K^2$
1	-1	1	1	-1	-1
2	3	16	0.25	1.5	12
5	24.6	625	0.04	4.92	615
8	26.6	642	1.29	5.42	626

2

$$A(642) + B(8) = 626$$
$$A(8) + B(1.29) = 5.42$$

using Cramer

$$A = \frac{\begin{vmatrix} 626 & 8 \\ 5.42 & 1.29 \end{vmatrix}}{\begin{vmatrix} 642 & 8 \\ 8 & 1.29 \end{vmatrix}} = \frac{764.18}{764.18} = 1$$

2

⇒ [illegible]

$$B = \frac{\begin{vmatrix} 642 & 626 \\ 8 & 5.42 \end{vmatrix}}{764.18} = \frac{-1528.36}{764.18} = -2$$

$$y = +x^2 - \frac{2}{x}$$

$$f(4) = +16 - \frac{2}{4} = \cancel{15.5}\ 15.5$$

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linearization: $y = ax^2 + \frac{b}{x}$

$$y = \frac{ax^3 + b}{x}$$

$$yx = ax^3 + b \qquad 2$$

$$Y = AX + B$$

$\Rightarrow$ $a = A$
$b = B$
$X = x^3$
$Y = yx$

x_i	y_i	X_i	Y_i	X_i^2	X_iY_i
1	-1	1	-1	1	-1
2	3	8	6	64	48
5	24.6	125	123	15625	15375
		134	128	15690	15422

2

$$A\Sigma X_i^2 + B\Sigma X_i = \Sigma X_iY_i$$

$$A\Sigma X_i + Bn = \Sigma Y_i$$

$$A(15690) + B(134) = 15422$$

$$A(134) + 3B = 128$$

using crammer

$$A = \frac{\begin{vmatrix} 15422 & 134 \\ 128 & 3 \end{vmatrix}}{\begin{vmatrix} 15690 & 134 \\ 134 & 3 \end{vmatrix}} = \frac{29114}{29114} = 1$$

2

$$B = \frac{\begin{vmatrix} 15690 & 15422 \\ 134 & 128 \end{vmatrix}}{29114} = \frac{-58228}{29114} = -2$$

$\Rightarrow$ يتبع صفحة

~~(c) natural spline~~

~~1   -1~~
~~2   3~~
~~5   24.6~~

~~$h_0 = 1$~~
~~$h_1 = 3$~~

~~$2(h_0 + h_1) = 6$~~

$$P_2' = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

3a2

1 $$E_2(X) = \frac{(X-X_0)(X-X_1)(X-X_2)\, f^{3}(c)}{6}$$

1 $$E_2'(X) = \frac{M_3}{6}\left[(X-X_0)(X-X_1) + (X-X_0)(X-X_2) + (X-X_1)(X-X_2)\right]$$

Backward $\rightarrow$ $E_2'(X_2)$ $\rightarrow$

1 $$E_2'(X_2) = \frac{M^3}{6}\left[\underbrace{(X_2-X_0)}_{(2h)}\underbrace{(X_2-X_1)}_{(h)}\right] = \frac{M^3}{6}2h^2 = \frac{M_3 h^2}{3}$$

Best h

3b

$$g = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{h^2 M_3}{3}$$

$$= \frac{3e + 4e + e}{2h} + \frac{h^2 M_3}{3}$$

$$g(h) = \frac{8e}{2h} + \frac{h^2 M_3}{3}$$ 1

$$= \frac{4e}{h} + \frac{h^2 M_3}{3}$$

$$g'(h) = \frac{-4e}{h^2} + \frac{2}{3} h M_3 = 0$$

$$\frac{4e}{h^2} = \frac{2}{3} h M_3$$ 1

$$\frac{6e}{M_3} = h^3$$

Best h $\rightarrow$ $h = \left(\frac{6e}{M_3}\right)^{\frac{1}{3}}$ 1

$f(x) = X^2 - \frac{2}{X}$

~~$f'(x) = 2x + 2$~~

$f(x) = X^2 - 2X^{-1}$

$f'(x) = 2X + 2X^{-2}$

$f''(x) = 2 - 4X^{-3}$

$f'''(x) = 12X^{-4} = \frac{12}{X^4}$

max $= \boxed{12}$

Best $h = \left(\frac{6(5\times10^{-9})}{12}\right)^{\frac{1}{3}}$

$= 1.357 \times 10^{-3}$

$$P_2'(x) = a_1 + a_2[(x-x_0)+(x-x_1)]$$

let $x_0 = x$, $x_1 = x-h$, $x_2 = x-2h$

(1) (3a1)

$$a_1 = F[x_0,x_1] = \frac{f(x_1)-f(x_0)}{-h}$$

$$= \frac{f(x_0)-f(x_1)}{h}$$

$$a_1 = F[x_0,x_1,x_2] = \frac{F[x_1,x_2]-F[x_0,x_1]}{-2h}$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{-h} - \dfrac{f(x_1)-f(x_0)}{-h}}{-2h}$$

(2)

$$= \frac{f(x_1)-f(x_2)+f(x_1)-f(x_0)}{(h)(-2h)}$$

$$= \frac{2f(x_1)-f(x_2)-f(x_0)}{-2h^2}$$

$$P_2'(x_0) = \frac{f(x_0)-f(x_1)}{h} + \left[\frac{2f(x_1)-f(x_2)-f(x_0)}{-2h^2}\right][0+h]$$

$$= \frac{2f(x_0)-2f(x_1)}{2h} + \frac{f(x_0)+f(x_2)-2f(x_1)}{2h}$$

$$= \frac{3f(x_0)-4f(x_1)+f(x_2)}{2h}$$

$$= \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$$

$$= \frac{3f_0-4f_{-1}+f_{-2}}{2h}$$

فرع (b) على الورقة الخارجية

Question 4. (3points).
Let $p_3(x)$ be the interpolating polynomial for the data $(0,0), (0.5, y), (1,3)$ and $(2,2)$. Find y if the coefficient of x^3 is 6.

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \cancel{\frac{(x-0.5)(x-1)(x-2)}{(-0.5)(-1)(-2)}} * 0 + \frac{(x-0)(x-1)(x-2)\,y}{(0.5)(-0.5)(-1.5)}$$

$$+ \frac{(x-0)(x-0.5)(x-2)(3)}{(1)(0.5)(-1)} + \frac{(x-0)(x-0.5)(x-1)*(2)}{(2)(1.5)(1)}$$

$$= x(x-1)(x-2)\,y\,(2.66667) + x(x-0.5)(x-2)(-6)$$
$$+ x(x-0.5)(x-1)(0.666667)$$

1

$$= (x^3-3x^2+2x)(2.66667)\,y + (x^3-2.5x^2+x)(-6)$$
$$+ (0.666667)(x^3-1.5x^2+0.5x)$$

$$2.66667 \quad (0.66667)x^3 = 6x^3$$

$$(2.66667)\,y - 6 + 0.66667 = 6$$

$$y = 4.249995$$

1