BIRZEIT UNIVERSITY

Key.

Mathematics Department

Math 330

Test 2

2^{nd} Semester 18-19

Student name: ………………………… ID no.: …… sec……..

Every problem 5 points

Q# 1) Consider the data

$(1, f(1)), (1.2, f(1.2))$ Where $f(x) = 2x^2 - \frac{1}{x}$

Find Lagrange interpolation polynomial $p_1(x)$ and use it to estimate $f(1.1)$

$(1, 1), (1.2, 2.05)$ with $x_0 = 1$, $y_0 = 1$, $x_1 = 1.2$, $y_1 = 2.05$

(3)
$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$
$$= \frac{x - 1.2}{-0.2}(1) + \frac{x - 1}{0.2}(2.05)$$
$$= -5(x - 1.2) + 10.25(x - 1)$$

(1) $f(1.1) \approx P_1(1.1) = -5(-0.1) + 10.25(0.1)$

$= 1.525 \approx 1.53$ (1)

Q#2) Find Newton interpolation polynomial $p_3(x)$ for $x \in [1, 2.5]$ that interpolate $f(x) = 2x^2 - \frac{1}{x}$, with equally spaced nodes x_0, x_1, x_2, x_3 and use it to estimate $f(1.1)$.

x_k	y_k	1st	2nd	3rd
1	1			
1.5	3.83	5.66		
2	7.5	7.34	1.68	
2.5	12.1	9.2	1.86	0.12

(4)

$a_0 = 1$, $a_1 = 5.66$, $a_2 = 1.68$, $a_3 = 0.12$

$P_1(x) = a_0 + a_1(x-1)$

$= 1 + 5.66(x-1)$

$P_1(1.1) = 1 + 5.66(0.1) = 1.57$

$P_2(x) = P_1(x) + a_2(x-1)(x-1.5)$

$P_2(1.1) = 1.57 + 1.68(0.1)(-0.4)$

$\approx +1.50$

$P_3(x) = P_2(x) + a_3(x-1)(x-1.5)(x-2)$

1

(1) $f(1.1) \approx P_3(1.1) = 1.50 + 0.12(0.1)(-0.4)(-0.9)$

≈ 1.5 (1)

Q#3) Find the best upper bound for the error $E_3(x)$ in (Q#2)

$$|E_3(x)| \le \frac{h^4 M_4}{24} \quad 2$$

$$\le \frac{(0.5)^4 \cdot (\cancel{24})}{(\cancel{24})} \quad 1$$

$$= 0.0625 \quad 1$$

$$f(x) = 2x^2 - \frac{1}{x}$$

$$f'(x) = 4x + \frac{1}{x^2}$$

$$f''(x) = 4 - \frac{2}{x^3}$$

$$f'''(x) = \frac{6}{x^4}$$

$$|f^{(4)}(x)| = \frac{24}{x^4}$$

$$M_4 = \max|f^{(4)}(x)|$$

$$1 \quad = \frac{24}{1} = 24$$

Q#4) Derive the normal equations for the best fitting curve of the form $f(x) = C\ln x + \frac{D}{x}$

$$E(C,D) = \sum_{k=1}^{n}\left(C\ln x_k + \frac{D}{x_k} - y_k\right)^2$$

$$\frac{\partial E}{\partial C} = 0 = \sum_{k=1}^{n} 2\left(C\ln x_k + \frac{D}{x_k} - y_k\right)\cdot \ln x_k$$

$$2.5 \quad C\sum_{k=1}^{n}(\ln x_k)^2 + D\sum_{k=1}^{n}\frac{\ln x_k}{x_k} = \sum_{k=1}^{n} y_k \ln x_k \quad \cdots (1)$$

$$\frac{\partial E}{\partial D} = 0 = \sum_{k=1}^{n} 2\left(C\ln x_k + \frac{D}{x_k} - y_k\right)\cdot \frac{1}{x_k}$$

$$2.5 \quad C\sum_{k=1}^{n}\frac{\ln x_k}{x_k} + D\sum_{k=1}^{n}\frac{1}{x_k^2} = \sum_{k=1}^{n}\frac{y_k}{x_k} \quad \cdots (2)$$

Q#5) Find C, D using the normal equations derived above (Q#4) and the given data.
$(2,0.87), (2.2,1.12), (2.4,-0.041)$

4

x_n	y_n	$\ln x_n$	$(\ln x_n)^2$	$\frac{\ln x_n}{x_n}$	$y_n \ln x_n$	$\frac{1}{x_n^2}$	$\frac{y_n}{x_n}$
2	0.87	0.69	0.48	0.35	0.60	0.25	0.44
2.2	1.12	0.79	0.62	0.36	0.88	0.21	0.51
2.4	−0.041	0.88	0.77	0.37	−0.036	0.17	−0.017
			1.87	1.08	1.44	0.63	0.93

$$1.87C + 1.08D = 1.44$$
$$1.08C + 0.63D = 0.93$$

1 $$C = \frac{\begin{vmatrix} 1.44 & 1.08 \\ 0.93 & 0.63 \end{vmatrix}}{\begin{vmatrix} 1.87 & 1.08 \\ 1.08 & 0.63 \end{vmatrix}} = \frac{-0.088}{0.012} \simeq -7.3$$

1 $$D = \frac{\begin{vmatrix} 1.87 & 1.44 \\ 1.08 & 0.93 \end{vmatrix}}{0.012} = \frac{0.18}{0.12} = 15$$

Q#6) Find a suitable Linearization for $f(x) = C\ln x + \frac{D}{x}$ (Don't find C, D)

$$y = C\ln x + \frac{D}{x}$$

3 $$\boxed{yx = Cx\ln x + D}$$
$$Y = AX + B$$

2 $$\left.\begin{array}{l} Y = yx \\ X = x\ln x \end{array}\right\} A, B$$
$$\Rightarrow C = A$$
$$D = B$$

or

3 $$\boxed{\frac{y}{\ln x} = \frac{D}{x\ln x} + C}$$
$$Y = AX + B$$

2 $$\left.\begin{array}{l} Y = \frac{y}{\ln x} \\ X = \frac{1}{x\ln x} \end{array}\right\} \Rightarrow A, B$$
$$D = A$$
$$B = C$$

Q#7) Derive the following formula using **Newton** interpolating Polynomial

$$f'(x_0) \cong \frac{f_1 - f_{-1}}{2h}$$

$$P_2(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1)$$

1 $$P_2'(t) = a_1 + a_2[(t-t_0)+(t-t_1)]$$

$$P_2'(t_0) = a_1 + a_2(t_0-t_1)$$

$$t_1 = t_0+h \quad , \quad t_2 = t_0-h$$

1 $$a_1 = f[t_0,t_1] = \frac{f(t_1)-f(t_0)}{t_1-t_0} = \frac{f(t_0+h)-f(t_0)}{h}$$

1 $$a_2 = f[t_0,t_1,t_2] = \frac{f[t_1,t_2]-f[t_0,t_1]}{t_2-t_0}$$

$$= \frac{\frac{f(t_0-h)-f(t_0+h)}{-2h} - \frac{f(t_0+h)-f(t_0)}{h}}{-h}$$

$$= \frac{f(t_0-h)-f(t_0+h)+2f(t_0+h)-2f(t_0)}{2h^2}$$

2 $$f'(x_0) \doteq P_2'(t_0) = \frac{f(t_0+h)-f(t_0)}{h} + \frac{f(t_0-h)+f(t_0+h)-2f(t_0)}{2h^2}(-h)$$

$$= \frac{2f(t_0+h)-2f(t_0)}{2h} - \frac{f(t_0-h)+f(t_0+h)-2f(t_0)}{2h}$$

$$= \frac{f(t_0+h)-f(t_0-h)}{2h}$$

Q#8) Estimate $f'(1.3)$ using the data (1.1,0.4238),(1.2, 1.003), (1.3,1.662),

How many digits of the answer do you expect to be correct.

3 $$f'(x_0) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

$$= \frac{3(1.662) - 4(1.003) + 0.4238}{2(0.1)} = 6.989$$

2 $$\text{Error} = c(0.1)^2 = c(0.01)$$

two digits accuracy.

Q#9) Find the optimal h for the following formula

$$f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_2}{12h^2} + \frac{h^4 f^{(6)}(c)}{90}$$

(1) $$g(h) = \frac{64\epsilon}{12h^2} + \frac{h^4 M}{90}$$

(1) $$g'(h) = -\frac{128\epsilon}{12h^3} + \frac{4h^3 M}{90} = 0$$

(1) $$\frac{2h^3 M}{45} = \frac{32\epsilon}{3h^3}$$

$$h^6 = \frac{(16)(\cancel{45}^{15})\epsilon}{\cancel{3}M} = \frac{240\epsilon}{M}$$

(2) $$h = \left(\frac{240\epsilon}{M}\right)^{1/6}$$

Q#10)Derive the truncation error of the following formula using the error associated to **Lagrange** interpolating Polynomial $f'(x_0) \cong \frac{-3f_0 + 4f_1 - f_2}{2h}$

(1) $$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

(2) $$E_2'(x) = \Big((x-x_1)(x-x_2) + (x-x_0)\big((x-x_1)+(x-x_2)\big)\Big)\frac{f^{(3)}(c)}{6}$$

$$E_2'(x_0) = \big((x_0-x_1)(x_0-x_2) + 0\big) \cdot \frac{f^{(3)}(c)}{6}$$

(2) $$= \frac{(-h)(-2h) \cdot f^{(3)}(c)}{6}$$

$$= \frac{h^2 f^{(3)}(c)}{3}$$