Q6, b

26/60

Birzeit University-Dept. of Mathematics
Maec330-Microeconomic Theory

Second Hour Exam (Instructor: Dr. Marwan Aloqeili) Spring 2011/2012
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Question 1 Find the largest domain over which the function

(a) $f(x,y) = x^2 - y^2 - xy - x^3$ is concave.

~~$f'(x,y)$ increasing $\Rightarrow$ $f_x = 2x - y - 3x^2 = 0$~~

~~$f_y = -2y - x$~~

~~$\Rightarrow f''(x,y) > 0$~~

~~$\Rightarrow f_{xx} = 2 - 6x = 0 \Rightarrow x = 3$~~

~~$f_{yy} = -2$~~

$f'(x,y)$ is decreasing $\Rightarrow$ $f''(x,y) \leq 0$ $\Rightarrow$ $f_x = 2x - y - 3x^2$

$f_{xx} = 2 - 6x$

$f_{xy} = -1$

$f_y = -2y - x$

$f_{yy} = -2$

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(b) $f(x,y) = \frac{x^2}{y}$ is convex.

$f_x = \frac{2x}{y} \Rightarrow f_{xx} = \frac{2}{y}$

$f_{xy} = \frac{-2x}{y^2}$, $f_y = -\frac{x^2}{y^2} \Rightarrow f_{yy} = \frac{+x^2 \cdot 2y}{y^4}$

$= \frac{2x^2}{y^3}$

$\Rightarrow D^2 f(x,y) = \begin{pmatrix} 2/y & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{pmatrix} \Rightarrow$ +ve def.

$\Rightarrow \frac{2}{y} > 0 \Rightarrow$ ~~$y > 0$~~ $\boxed{y > 0}$

$\Rightarrow \frac{4x^2}{y^4} - \frac{4x^2}{y^4} = 0$ ~~$\Rightarrow x = y \Rightarrow x > 0$~~

~~$\Rightarrow f(x,y)$ convex on $C^+_{(x>0,\, y>0)}$~~

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Question 2 Prove the following:

(a) If $f(x,y)$ is homogeneous of degree one then $x^2 f_{xx} = y^2 f_{yy}$.

$K = 1$

$$x \frac{\partial}{\partial x} f(x,y) + y \frac{\partial}{\partial y} f(x,y) = K f(x,y) \quad \text{(by Thm)}$$

$$\Rightarrow x f_x + y f_y = 1 f(x,y)$$

$$\Rightarrow xx f_{xx} + yy f_{yy} = (K-1) K f(x,y)$$

$$\Rightarrow xx f_{xx} + yy f_{yy} = (1-1) 1 f(x,y)$$

$$\Rightarrow x^2 f_{xx} + y^2 f_{yy} = 0 \Rightarrow x^2 f_{xx} = -y^2 f_{yy}$$

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(b) The function $f(x,y) = x^9y^3 + x^3y + 1$ is homothetic.

we need to show that $\dfrac{\frac{\partial}{\partial x} f(x,y)}{\frac{\partial}{\partial y} f(x,y)} = \dfrac{\frac{\partial}{\partial x} f(tx,ty)}{\frac{\partial}{\partial y} f(tx,ty)}$

Let $f(x,y) = g(u(x,y))$ ⟶ 0

$u(x,y) = x^3 y$

$g(z) = z^3 + z + 1 \Rightarrow g(z)$ is not Homog. ⇒ not Homoth.

0

(c) If $f(\mathrm{x})$ is homogeneous of degree k and $g(\mathrm{x})$ is homogeneous of degree l then $h(\mathrm{x}) = f(\mathrm{x})g(\mathrm{x})$ is homogeneous of degree $k + l$.

Let $f(x)$ is homog. of degree K.

$\Rightarrow f(tx) = t^K f(x)$ ---- (1)

and Let $g(x)$ is homog. of degree L

$\Rightarrow g(tx) = t^L g(x)$ ---- (2)

$\Rightarrow h(x) = f(x) g(x)$

$\Rightarrow h(tx) = f(tx) g(tx)$ by substitute (1) and (2)

$\Rightarrow h(tx) = t^K f(x) \cdot t^L g(x)$

$= t^{K+L} f(x) g(x) \Rightarrow h(tx) = t^{K+L} h(x)$

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Question 3 Consider the problem of optimizing $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 3$. Use the second order conditions to show that the point $(x, y) = (\sqrt{3}, -\sqrt{3}), \lambda = 2$ is a maximum. Then, use the envelope theorem to estimate the maximum of f subject to the constraint $1.1x^2 + xy + y^2 = 3$.

$$L(x, y, \mu) = x^2 + y^2 - \mu\,(x^2 + xy + y^2 - 3)$$

$$\Rightarrow L_x = 2x - 2\mu x - \mu y$$

$$L_y = 2y - 2\mu y - \mu x$$

$$L_\mu \Rightarrow \quad x^2 + xy + y^2 = 3.$$

$$L_{xx} = 2 - 2\mu \qquad h_x = 2x + y \qquad h_y = x + 2y$$

$$L_{xy} = -\mu$$

$$L_{yy} = 2 - 2\mu$$

$$\Rightarrow D^2 L(x, y, \mu) = \begin{pmatrix} 0 & 2x+y & x+2y \\ 2x+y & 2-2\mu & -\mu \\ x+2y & -\mu & 2-2\mu \end{pmatrix}$$

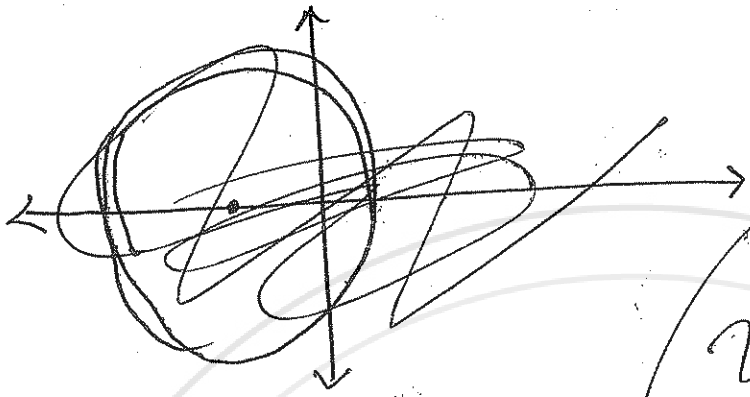
$$\Rightarrow D^2 L(\sqrt{3}, -\sqrt{3}, 2) = \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -2 & -2 \\ -\sqrt{3} & -2 & -2 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -2 & -2 \\ -\sqrt{3} & -2 & -2 \end{pmatrix}^3 = -\sqrt{3}\left(-2\sqrt{3} - 2\sqrt{3}\right) - \sqrt{3}\left(-2\sqrt{3} - 2\sqrt{3}\right)$$

$$= -\sqrt{3}\left(-4\sqrt{3}\right) - \sqrt{3}\left(-4\sqrt{3}\right)$$

Question 4 Consider the problem of minimizing $f(x, y) = (x+1)^2 + y^2$ subject to the constraint $y^2 - x^3 = 0$.

(a) Solve the above problem graphically.



$y^2 = x^3$

$y = x^{3/2} = \sqrt[2]{x^3}$

2 → 0/0

(b) Solve the problem by including a multiplier μ_0 for the objective function.

$L(x, y, \mu_0, \mu_1) = \mu_0((x+1)^2 + y^2) - \mu_1(y^2 - x^3)$

$L_x = \mu_0 2(x+1) + 3\mu_1 x^2 = 0 \Rightarrow$

$L_y = \mu_0 2y - 2\mu_1 y = 0$

$y^2 - x^3 = 0 \Rightarrow y^2 = x^3$

$\mu_0 = 0$ or 1 ~~not~~ both μ_1 and μ_2 equal zero.

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$\cancel{2(x+1) + 3\mu_1 x^2 = 0 \Rightarrow 3\mu_1 x^2}$

$2\mu_0(x+1) + 3\mu_1 x^2 = 0$

$2\mu_0 y - 2\mu_1 y = 0 \Rightarrow \mu_0 = \mu_1$ if $y \neq 0$.

$\Rightarrow \mu_0 = \mu_1 \neq 0$ if $y \neq 0 \Rightarrow x \neq 0$.

$\Rightarrow \mu_0(2x+2) + 3\mu_0 x^2 = 0 \Rightarrow 3\mu_0 x^2 + 2\mu_0 x + 2\mu_0 = 0$

$\Rightarrow$

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Question 5 Determine whether the following functions are quasiconvex, quasiconcave, or neither

(a) $f(x,y) = e^{-x^2-y^2}$.

(b) $f(x,y) = ye^x, y < 0$.

(c) $f(x,y,z) = \sqrt{\sqrt{x} + \sqrt{y} + \sqrt{z}}$.

zero

Question 6 Prove the following:

(a) If f is convex then $\text{epi}(f) = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} | f(\mathbf{x}) \leq y\}$ is a convex set.

f is convex ⇒ f is defined on convex set say U

⇒ $f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$

(b) Suppose that $f(\mathbf{x})$ is defined on an open convex set $S \subset \mathbb{R}^n$ and $g(u)$ is defined over an interval in $\mathbb{R}$ that contains $f(\mathbf{x})$, $\forall \mathbf{x} \in S$. Show that if $f(\mathbf{x})$ is convex, $g(u)$ is convex and increasing then $h(\mathbf{x}) = g(f(\mathbf{x}))$ is convex.

given g(u) is convex and increasing

⇒ $g'(u) > 0$ ⇒ its strictly increasing.

⇒ g(u) is M.T (Mon. Trans)

⇒ since f(x) is ~~convex~~ ⇒ f(x) is homog. function ~~[illegible]~~ ⇒ g(f(x)) is homoth.

and since the homoth. function is ordinar ⇒ g(f(x)) is homoth. func. and convex.