# Newton's Law of Cooling

States that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). That is, $\boxed{\frac{du}{dt} = -k(u-T)}$,

where $T$ is the constant ambient temp.

and $k$ is a positive constant.

$u(t)$ is the temperature of an object at any time $t$.

Ex. Spse that the temperature of a cup of coffee obeys Newton's Law of Cooling. If the coffee has a temperature of 90°c when freshly poured, and 1 min Later has cooled to 85°c in a room at 20°c, determine when the coffee reaches a temperature of 65°c.

Sol.

$$\frac{du}{dt} = -k(u-T), \quad u(0)=90, \quad u(1)=85$$

$$T = 20^\circ c, \quad u(t) = 65$$
$$t = ??$$

Sol.

$$\frac{du}{dt} = -k(u-20) \Rightarrow \int \frac{du}{u-20} = -k\int dt$$

$$\Rightarrow \ln|u-20| = -kt + c$$

$$u-20 = \underbrace{\pm e^{c}}_{A} e^{-kt}$$

$$\boxed{u(t) = 20 + Ae^{-kt}}$$

$$u(0) = 90 \Rightarrow 20 + A = 90 \Rightarrow \boxed{A = 70}$$

$$u(1) = 85 \Rightarrow 20 + 70e^{-k} = 85$$

$$70e^{-k} = 65 \Rightarrow \boxed{k = -\ln\left(\frac{65}{70}\right)}$$

$$u(t) = 20 + 70e^{\ln\left(\frac{65}{70}\right)t} = 20 + 70\,e^{\ln\left(\frac{65}{70}\right)^t}$$

$$\boxed{u(t) = 20 + 70\left(\frac{65}{70}\right)^t}$$

22. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

$\frac{dV}{dt} \propto S$, where $V = \text{Volume} = \frac{4}{3}\pi r^3$

$S = \text{Surface Area} = 4\pi r^2$

$\Rightarrow \frac{dV}{dt} = -kS, \ k>0$

$V = \frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3}{4\pi}V\right)^{\frac{1}{3}}$

$\therefore S = 4\pi\left(\frac{3}{4\pi}V\right)^{\frac{2}{3}}$

$$\frac{dV}{dt} = -\underbrace{k \cdot 4\pi\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}}_{\beta} V^{\frac{2}{3}}$$

$$\frac{dV}{dt} = -\beta V^{\frac{2}{3}}, \ \beta > 0.$$

23. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is 0.05 $(\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

$T = 70°F \qquad k = 0.05\,(\text{min})^{-1}$

$$\frac{du}{dt} = -k(u - T)$$

$$\frac{du}{dt} = -0.05(u - 70),$$

$u(t)$ in °F

$t$ in min.

24. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm$^3$ of the drug enters the patient's bloodstream at a rate of 100 cm$^3$/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time?

Let $q(t)$ be the amount of the drug at any time $t$.

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$= (5)(100) - 0.4q$$

$$\therefore \boxed{\frac{dq}{dt} = 500 - 0.4q}$$

(a) $\displaystyle\int \frac{dq}{500-0.4q} = \int dt \Rightarrow \int \frac{-0.4\,dq}{500-0.4q} = \int -0.4\,dt$.

$$\ln|500-0.4q| = -0.4t + C$$

$$500 - 0.4q = \pm e^{-0.4t+C} = \underbrace{\pm e^{C}}_{A} e^{-0.4t}$$

$$\Rightarrow 0.4q = 500 - Ae^{-0.4t}.$$

$$\therefore \boxed{q(t) = 1250 + Be^{-0.4t}}, \quad B = \frac{-A}{0.4}$$

$$\lim_{t\to\infty} q(t) = 1250 + 0 = 1250.$$

11. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

(a) If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$\boxed{dv/dt = [(49)^2 - v^2]/245.}$$

Also see Problem 25 of Section 1.1.

(b) If $v(0) = 0$, find an expression for $v(t)$ at any time.

(c) Plot your solution from part (b) and the solution (26) from Example 2 on the same axes.

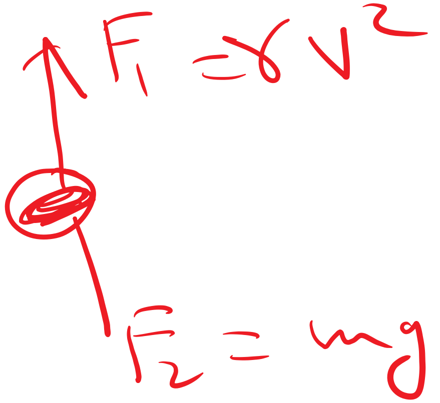
(d) Based on your plots in part (c), compare the effect of a quadratic drag force with that of a linear drag force.

$x = \int v(t)\,dt$

(e) Find the distance $x(t)$ that the object falls in time $t$. $x(0) = 0$

(f) Find the time $T$ it takes the object to fall 300 m.

$$F_{net} = F_2 - F_1 \Rightarrow ma = mg - \gamma v^2$$




$$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m} v^2}$$

ⓐ $V_L = \lim_{t\to\infty} V(t) = 49,$ $m = 10.$

$$\frac{dV}{dt} = 9.8 - \frac{\gamma}{10} V^2$$

$$V_L = 49 \Rightarrow 0 = 9.8 - \frac{\gamma}{10} V_L^2$$

$$0 = 9.8 - \frac{\gamma}{10}(49)^2$$

$$\Rightarrow \gamma = 9.8 \frac{10}{(49)^2} = \frac{98}{(49)^2} = \frac{2}{49}.$$

$$\frac{dv}{dt} = 9.8 - \frac{\frac{2}{49}}{10} v^2 = \frac{49}{5} - \frac{1}{(49)(5)} v^2$$

$$= \frac{(49)^2 - v^2}{(49)(5)}$$

$$\rightarrow \boxed{\frac{dv}{dt} = \frac{(49)^2 - v^2}{245}}$$

(b) $v(0) = 0$ find $v(t)$ $\int \frac{dv}{(49)^2 - v^2} = \int \frac{1}{245} dt$

12. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time $t$, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate $r$.

(b) Find an expression for the amount of thorium-234 present at any time $t$.

(c) Find the time required for the thorium-234 to decay to one-half its original amount. $Q(t) = 50$

$t$

$$\frac{dQ}{dt} = -rQ, \quad r>0$$

a

$$\begin{cases} Q(0) = 100 \\ Q(7) = 82.04 \end{cases}$$

find $r$.

Sol. $Q(0) = \boxed{A = 100}$

$Q(7) = 100e^{-7r} = 82.04$

b

$$\int \frac{dQ}{Q} = -r\int dt$$

$$\ln|Q(t)| = -rt + c$$

$$\boxed{Q(t) = Ae^{-rt}}$$

$$e^{-7r} = \frac{82.04}{100} \Rightarrow \boxed{r = -\frac{1}{7}\ln\left(\frac{82.04}{100}\right)}.$$

(c) $$Q(t) = 50 \Rightarrow 100e^{\frac{1}{7}\ln\left(\frac{82.04}{100}\right)t} = 50$$

$$\left(\frac{82.04}{100}\right)^{\frac{t}{7}} = \frac{1}{2}$$

$$\boxed{t = \frac{7\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{82.04}{100}\right)}} \approx .$$

15. According to Newton's law of cooling (see Problem 23 of Section 1.1), the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T),$$

where $T$ is the constant ambient temperature and $k$ is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

(a) Find the temperature of the object at any time.

(b) Let $\tau$ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between $k$ and $\tau$.

(a) $\dfrac{du}{dt} + \underbrace{k}_{p(t)}u = \underbrace{kT}_{g(t)}$, linear in $u$

$$\mu(t) = e^{\int k\,dt} = e^{kt}.$$

$$u(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + C\right]$$

$$= \frac{1}{e^{kt}}\left[\int e^{kt} \cdot kT\, dt + C\right]$$

$$= e^{-kt}\left[\frac{e^{kt} \cdot \cancel{k}T}{\cancel{k}} + C\right] = T + C e^{-kt}.$$

$$u_0 = u(0) = T + C \Rightarrow \boxed{C = u_0 - T}.$$

$$\therefore \boxed{u(t) = T + (u_0 - T)\, e^{-kt}}$$

(b) $u(\tau) = T + (u_0 - T)e^{-k\tau}$

$$\underbrace{u(\tau) - T}_{\text{Given}} = (u_0 - T)e^{-k\tau}$$

$$\frac{1}{2}\cancel{(u_0 - T)} = \cancel{(u_0 - T)}e^{-k\tau}$$

$$\ln\left(\frac{1}{2}\right) = -k\tau$$

$$\therefore \boxed{k\tau = \ln 2}$$

16. Suppose that a building loses heat in accordance with Newton's law of cooling (see Problem 15) and that the rate constant $k$ has the value $0.15\,\mathrm{h}^{-1}$. Assume that the interior temperature is 70°F when the heating system fails. If the external temperature is 10°F, how long will it take for the interior temperature to fall to 32°F?

$k$

$u(0)$

$T$

$$\frac{du}{dt} = -k(u-T), \text{ where } k = 0.15, \; T = 10°F$$

$$u(0) = 70°F$$

We need to find $t$ such that $u(t) = 32°$.

$$\frac{du}{dt} = -0.15(u-10)$$

$$\int \frac{du}{u-10} = \int -0.15\, dt \Rightarrow \ln|u-10| = -0.15t + C$$

$$u(t) - 10 = A e^{-0.15t} \Rightarrow \boxed{u(t) = 10 + A e^{-0.15t}}$$

$$70 = u(0) = 10 + A \Rightarrow \boxed{A = 60}$$

$$\therefore \boxed{u(t) = 10 + 60 e^{-0.15t}}$$

$$32 = 10 + 60 e^{-0.15t}$$

$$\frac{22}{60} = e^{-0.15t} \Rightarrow -0.15t = \ln\left(\frac{11}{30}\right)$$

$$\therefore t = \frac{100}{15} \ln\left(\frac{30}{11}\right) \approx 6.7\,h$$

$Q(0) = 0$

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of $\gamma$ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of $\gamma$ for the amount of salt in the tank at any time $t$. Also find the limiting amount of salt in the tank as $t \to \infty$.

Let $Q(t)$ be the amount of salt in the tank at any time $t$.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= (\gamma)(2) - \frac{Q(t)}{120}(2), \quad Q(0) = 0.$$

$$\frac{dQ}{dt} + \underbrace{\frac{1}{60}}_{p(t)}Q = \underbrace{2\gamma}_{g(t)}, \quad Q(0) = 0 \qquad \text{linear in } Q.$$

$$\mu(t) = e^{\int \frac{1}{60} dt} = e^{\frac{1}{60}t}$$

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\, dt + c\right]$$

$$= e^{-\frac{1}{60}t}\left[\int e^{\frac{1}{60}t} \cdot 2\gamma\, dt + c\right]$$

$$= e^{-\frac{1}{60}t}\left[120\gamma\, e^{\frac{1}{60}t} + c\right]$$

$$\boxed{Q(t) = 120\gamma + c e^{-\frac{1}{60}t}}$$

$$0 = Q(0) = 120\gamma + c \Rightarrow \boxed{c = -120\gamma}$$

$$\therefore \quad Q(t) = 1208 - 1208\, e^{-\frac{1}{60}t} = 1208\left(1 - e^{-\frac{t}{60}}\right).$$

$$Q_L = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(1208 - 1208\, e^{-\frac{1}{60}t}\right)$$

$$= 1208 - 0$$

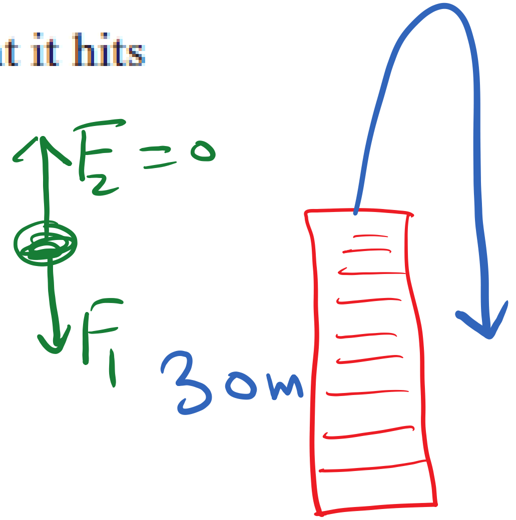
$$= 1208.$$

20. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

(a) Find the maximum height above the ground that the ball reaches.

(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.

(c) Plot the graphs of velocity and position versus time.




Solution. $m = 0.15\ \text{kg}$, $v(0) = 20\ \text{m/s}$.

$$F_{net} = F_2 - F_1$$

$$ma = 0 - mg \Rightarrow \boxed{\frac{dv}{dt} = -9.8}$$

$$\int dv = -\int 9.8\,dt \Rightarrow \boxed{v(t) = -9.8t + C}$$

$$v(0) = 20 \Rightarrow \boxed{C = 20}$$

$\therefore \boxed{v(t) = -9.8t + 20}$

ⓐ $x(t) = \int v(t)\,dt = \int (-9.8t + 20)\,dt$

$$\boxed{x(t) = -4.9t^2 + 20t + k}$$

Notice that $x(0) = 30\text{m} \Rightarrow \boxed{k = 30}$

$$\boxed{x(t) = -4.9t^2 + 20t + 30}$$

Maximum height ?? $v(t) = 0 \Rightarrow -9.8t + 20 = 0$

$t = \frac{20}{9.8}$

Maximum height $= x\left(\frac{20}{9.8}\right) = -4.9\left(\frac{20}{9.8}\right)^2 + 20\left(\frac{20}{9.8}\right) + 30$

$= \ldots$

(b) the time when the ball hits the ground

means $x(t) = 0$

$$\Rightarrow -4.9t^2 + 20t + 30 = 0$$

$$t = \frac{-20 \pm \sqrt{(20)^2 - 4(-4.9)(30)}}{2(-4.9)} = \ldots$$