BIRZEIT UNIVERSITY

Mathematics Department

Ordinary Differential Equations – Math331

First Exam First Semester 2019 – 2020

Name: Key A Number: ____________ Section: ____________

Section	Instructor	Day	Time
1	Ala Talahmeh	SM	10:00 - 11:15
2	Alaeddin Elayyan	SM	11:25 - 12:40
3	Muna Abu Alhalawa	TR	10:00 - 11:15
4	Abdelrahim Mousa	TR	12:50 - 14:05
5	Ala Talahmeh	TR	11:25 - 12:40
6	Abdelrahim Mousa	MW	10:00 - 11:15
7	Alaeddin Elayyan	SW	12:50 - 14:05

Question One (26 points) Circle the most correct answer:

1. The differential equation $\frac{dy}{dx} - y^2 + y = 0$ is

   (a) not exact

   (b) separable

   (c) Bernoulli

   $\boxed{\textbf{(d)}}$ All of the above

2. The DE $2xy\, dx + \left(x^2 - y^2\right) dy = 0$ is

   $\boxed{\textbf{(a)}}$ exact

   (b) linear

   (c) separable

   (d) exact and separable

3. Suppose the change on a temperature of a hot cup obeys to Newton's law of cooling. The hot cup initially has a temperature of $100F$ and brought to a room with temperature $30F$. If the temperature of the cup becomes $65F$ in $\ln 2$ minutes, then the temperature of the cup after $\ln 7$ minutes is

   (a) $50F$

   (b) $20F$

   $\boxed{\textbf{(c)}}$ $40F$

   (d) $30F$

4. The integrating factor of the DE $(x^2 - xy)y' = y^2 - 3xy$, $x > 0$ is

(a) y
(b) x
(c) xy
(d) None of the above

5. The DE $ty' - y \ln\frac{y}{t} = 5t, t > 0$ is

(a) 1^{st} order, linear, homogeneous ODE
(b) 1^{st} order, linear, non-homogeneous ODE
(c) 1^{st} order, non-linear, non-homogeneous ODE
(d) 1^{st} order, non-linear, homogeneous ODE

6. The IVP: $\frac{dx}{dt} = \sqrt{x}, x(0) = 1$

(a) has more than two solution
(b) has a unique solution
(c) has no solution
(d) has two solutions

7. If $y' - 2y = e^t$, $y(0) = 2$, then $y(\ln 2) =$

(a) 0
(b) 2
(c) 5
(d) 10

8. The half-life time of a radioactive material is $\ln 8$ years. After $\ln 64$ years, this material becomes 10 gm. The initial amount of this material is

(a) 32 gm
(b) 40 gm
(c) 24 gm
(d) 12 gm

9. If $y(x)$ solves the IVP: $y' - y^2 + y = 0,\ y(0) = -1$, then, $\lim\limits_{x\to\infty} y(x) =$

(a) 0
(b) -1
(c) 1
(d) $-\infty$

10. The behavior of solution diverges in the DE

(a) $y' = 1 + y$
(b) $y' = 1 - y$
(c) $y' + 2y = -2$
(d) $y' + 3 = -y$

11. The largest interval in which the solution of the IVP $(\ln t)y' + \frac{1}{(t-3)}y = t^2,\ y(2) = 4$ is defined

(a) $(1,3)$
(b) $(3,\infty)$
(c) $(0,3)$
(d) $(0,\infty)$

12. If $y(t)$ solves the DE $y' - t^3 y = y^2$, then the slope of $y(t)$ at the point $(1,2)$ is

(a) 2
(b) 4
(c) 6
(d) 0

13. The solution of the IVP $y' - 3y - 9 = 0$, $y(0) = 1$ is

(a) $y(t) = e^{3t}$
(b) $y(t) = 2e^{3t} - 1$
(c) $y(t) = 3e^{3t} - 2$
(d) $y(t) = 4e^{3t} - 3$

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Question Two (7 points) Find an explicit solution for the IVP:

$$\frac{dy}{dx} = \frac{y(x-y)}{x^2}\,, \qquad y(1) = -1\,, \qquad x > 0$$

Homogenuous

1 $$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

1 Let $V = \frac{y}{x} \Rightarrow y' = xV' + V$

1 $$xV' + \cancel{V} = \cancel{V} - V^2 \Rightarrow -V^{-2}\,dV = \frac{dx}{x}$$

1 $$\frac{1}{V} = \ln x + C \Rightarrow V = \frac{1}{\ln x + C}$$

1 $$\frac{y}{x} = \frac{1}{\ln x + c}$$

1 $$c = -1$$

1 $$y(x) = \frac{x}{\ln x - 1}$$

Question Two (7 points) Find an explicit solution for the IVP:

Benoulli

$$\frac{dy}{dx} = \frac{y(x-y)}{x^2}, \qquad y(1) = -1, \qquad x > 0$$

1 $$y' - \frac{1}{x} y = \frac{-1}{x^2} y^2$$

$n=2$, $p(x) = \frac{-1}{x}$, $g(x) = \frac{-1}{x^2}$

1 $$V = y^{1-n} = \frac{1}{y}$$

1 $$\begin{cases} V' + (1-n) p(x) V = (1-n) g(x) \\ V' + \frac{1}{x} V = \frac{1}{x^2} \end{cases}$$

1 $$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

1 $$\begin{cases} V(x) = \frac{1}{\mu}\left[\int \mu(x) g(x) dx + c\right] \\ \quad = \frac{1}{x}\left[\ln x + c\right] \end{cases}$$

1 $$\begin{cases} \frac{1}{y} = \frac{\ln x + c}{x} \\ y(x) = \frac{x}{\ln x + c} \end{cases}$$

1 $$c = -1$$

Question Three (8 points) Solve the IVP:

$$xy^3 + (x^2y^2+1)y' = 0, \qquad y(2) = 1\,, \qquad x > 0, \qquad y > 0$$

(1) $\begin{cases} M = xy^3 \Rightarrow M_y = 3xy^2 \\ N = x^2y^2+1 \Rightarrow N_x = 2xy^2 \end{cases}$ not exact

(1) $\dfrac{M_y - N_x}{M} = \dfrac{3xy^2 - 2xy^2}{xy^3} = \dfrac{1}{y}$

(1) $I(y) = e^{-\int \frac{1}{y}dy} = e^{-\ln y} = \dfrac{1}{y}$

Multiply by $\frac{1}{y}$ $\Rightarrow$

(1) $xy^2 + \left(x^2y + \frac{1}{y}\right)y' = 0$

$M = xy^2 \Rightarrow M_y = 2xy$

$N = x^2y + \frac{1}{y} \Rightarrow N_x = 2xy$ Exact

(1) $\psi = \int \psi_x \, dx = \int M\,dx = \int xy^2 dx = \frac{x^2}{2}y^2 + g(y)$

(1) $\psi_y = \cancel{x^2y} + g'(y) = N = \cancel{x^2y} + \frac{1}{y} \Leftrightarrow g(y) = \ln y$

(1) $\frac{x^2}{2}y^2 + \ln y = C$

(1) $C = 2$

Question Three (8 points) Solve the IVP:

$$xy^3 + (x^2y^2+1)y' = 0, \qquad y(2) = 1\,, \qquad x>0, \qquad y>0$$

$$xy^3\,dx + (x^2y^2+1)\,dy = 0$$

1 $$\frac{dx}{dy} + \frac{1}{y}x = -\frac{1}{y^3}x^{-1}$$ Bernoulli with $n=-1$

$p(y) = \frac{1}{y}$

$q(y) = -\frac{1}{y^3}$

1 $$V = x^{1-n} = x^2$$

$$V' + (1-n)p(y)V = (1-n)q(y)$$

1 $$V' + \frac{2}{y}V = \frac{-2}{y^3}$$

1 $$\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$$

1 $$V(y) = \frac{1}{y^2}\left[\int y^2\left(\frac{-2}{y^3}\right)dy + c\right]$$

1 $$= \frac{1}{y^2}\left[c - 2\ln y\right]$$

1 $$x^2 = \frac{1}{y^2}\left[c - 2\ln y\right]$$

$$x^2y^2 + 2\ln y = c$$

$$c = 4$$

1

<u>Question Four</u> (9 points) Given the IVP:

$$y' + \frac{4}{x}y = x^3y^2, \qquad y(1) = \frac{1}{2}, \qquad x > 0$$

(1) Find the unique solution

(2) Find the interval where the solution is defined

Bernoulli

$n = 2, \quad p(x) = \frac{4}{x}, \quad q(x) = x^3$

1 $\quad V = y^{1-n} = \frac{1}{y}$

1 $\quad \begin{cases} V' + (1-n)\,p(x)\,V = (1-n)\,q(x) \\ V' - \frac{4}{x}V = -x^3 \end{cases}$

1 $\quad \mu(x) = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = \frac{1}{x^4}$

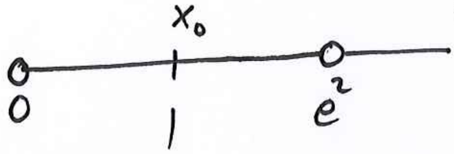
1 $\quad \begin{cases} V(x) = \frac{1}{\mu}\left[\int \mu g\,dx + c\right] \\ \quad\;\; = x^4\left[-\ln x + c\right] \end{cases}$

1 $\quad \frac{1}{y} = x^4\left[c - \ln x\right]$

1 $\quad c = 2$

1 $\quad y(x) = \frac{1}{x^4\left[2 - \ln x\right]}$

1 $\quad 2 - \ln x \neq 0 \iff x \neq e^2$



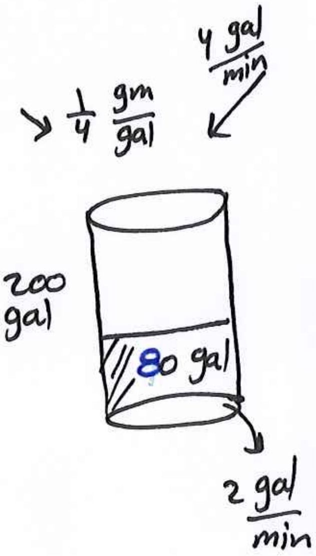
1 $\quad I = (0, e^2)$

Question Five (10 points) A tank of capacity 200 gal has initially 0.1 gm of toxic wastes dissolved in 80 gal of water. Water with toxic wastes starts flow into the tank at rate 4 gal/min and flow out at rate 2 gal/min. The incoming water contains $\frac{1}{4}$ gm/gal of toxic wastes.

(a) Write IVP that models the change of toxic wastes in the tank over time.

(b) Find the amount of toxic wastes in the tank at any time.

(c) Find the amount of toxic wastes in the tank when it becomes to overflow.



$\boxed{a}$ $\dfrac{dQ}{dt} = \left(\frac{1}{4}\right)(4) - (2)\,\dfrac{Q}{80+2t}$, $Q(0) = 0.1$

2 + 1 $\quad = 1 - \dfrac{2}{80+2t}\,Q$, $Q_0 = 0.1$

1 $\boxed{b}$ $Q' + \dfrac{2}{80+2t}\,Q = 1$

1 $\mu = e^{\int \frac{2\,dt}{80+2t}} = e^{\ln(80+2t)} = 80 + 2t$

1 $Q = \dfrac{1}{80+2t}\left[\int (80+2t)\,dt + c\right]$

1 $\quad = \dfrac{1}{80+2t}\left[80t + t^2 + c\right]$

1 $c = 8$

1 $\boxed{c}$ The tank overflows after $t = 60$ min

$Q(60) = \dfrac{1}{80+2(60)}\left[80(60) + (60)^2 + 8\right]$

1 $\quad = \dfrac{8408}{200}$

$\quad = 42{,}04$ gm

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(2 + (1 [a] $Q' = 1 - \frac{Q}{40+t}, \quad Q_0 = 0.1$

(1 [b] $Q' + \frac{1}{40+t} Q = 1$

(1 $\mu = e^{\int p(t)dt} = e^{\int \frac{dt}{40+t}} = e^{\ln(40+t)} = 40+t$

(1 $Q = \frac{1}{40+t}\left[\int (40+t)dt + c\right]$

(1 $= \frac{1}{40+t}\left[40t + \frac{t^2}{2} + c\right]$

(1 $c = 4$

(1 [c] The tank overflows after $t = 60$ min

$Q(60) = \frac{1}{40+60}\left[40(60) + \frac{1}{2}(60)^2 + 4\right]$

(1 $= 24 + 18 + \frac{4}{100}$

$= 42.04$ gm

Good Luck