Question 1(30 points) Circle the correct answer

(1) Suppose that the limiting velocity of a falling object, whose mass is 5 kg, is $7m/s$. If the drag force is γv then $\gamma =$

(a) 7 kg/s
(b) 49 kg/s
(c) 14 kg/s
(d) 32 kg/s

$v'(t) = \frac{mg - \gamma v}{m}$

$0 = mg - \gamma v$

$49 - \gamma \cdot 7$

$\boxed{\gamma = 7}$

mg

(2) Any solution of the differential equation $(x^2 + y^2) + (2xy)y' = 0$ satisfies the equation $\psi(x, y) = c$. Then

(a) $\psi(x, y) = x^3 + xy^2$
(b) $\psi(x, y) = \frac{1}{3}x^3 + xy^2 - y$
(c) $\psi(x, y) = \frac{1}{3}x^3 + xy^2 + y$
(d) $\psi(x, y) = \frac{1}{3}x^3 + xy^2$

$2y \quad 2y$

$\boxed{\frac{x^3}{3} + xy^2 + C}$

$\boxed{2xy} + h'(y) = 0$

(3) Using Picard's method to solve the I.V.P. $y' = ty - 1$, $y(0) = 0$ we find

(a) $\phi_2(t) = t^3 - t$
(b) $\phi_2(t) = -\frac{1}{3}t^3 - 1$
(c) $\phi_2(t) = -\frac{1}{3}t^3 - t$
(d) $\phi_2(t) = \frac{1}{3}t^3 - t$

$\phi_0 = 0$

$\phi_1 = \int_0^t f(s, \phi_0)\, ds$

$= \int_0^t s\phi_0 - 1\, ds = -t$

$-s^2 - 1 \quad ds$

$\boxed{-\frac{1}{3}s^3 - s}$

(4) The solution of the diff. equation $y' - \frac{y}{x} = e^{y/x}$, $x > 0$ satisfies the equation

(a) $e^{-y/x} = \ln x + c$
(b) $e^{y/x} = \ln x + c$
(c) $e^{-y/x} = -\ln x + c$
(d) $e^{y/x} = -\ln x + c$

$\frac{y}{x} = v(x)$

$-e$

$v + x\frac{dv}{dx} - v = e^v$

$\frac{dy}{dt} = \frac{dy}{dv}\frac{dv}{dt}$

$17 \times 1.5 = 25.5$

1

$y = xv$

$\frac{dy}{dx} = \frac{dx}{dv} \cdots + \cdots x$

$\frac{dv}{e^v} = \frac{dx}{x}$

$-e^{-v} = -\ln x +$

(5) If $\lim_{t\to\infty} y(t) = 2$ for any solution $y(t)$ of the differential equation $y'(t) = ay + 2$, then

$0 = ay + 2$

$ay =$

$\underline{a = -1}$

(a) $a = 1$

(b) $a = -1$

(c) $a = 2k$ for any constant k

(d) there is no value for a

(6) Let $y(t)$ be the solution of the I.V.P. $y'' + 2y' - 3y = 0$, $y(0) = 1, y'(0) = \alpha$. The value of α for which $\lim_{t\to\infty} y(t) = 0$ is

(a) 1

(b) -1

(c) 3

(d) -3

$y(t) = c_1 e^{t} + c_2 e^{-3t}$

$y(t) \to 0$

$\boxed{c_1 = 0}$

$r^2 + 2r - 3 = 0$

$(r-1)(r+3) = 0$

$y_1 = e^{t}$ $\quad y_2 = e^{-3t}$

$c_1 + c_2 = 1 \rightarrow c_2 = 1$

$c_1 - 3c_2 = \alpha$

$\boxed{\alpha = -3}$

(7) The differential equation $w\frac{dz}{dw} + \frac{z^2}{w} = z$ is

(a) First order linear differential equation.

(b) First order exact equation.

(c) First order separable equation.

(d) First order homogeneous equation.

$\frac{dz}{dw} + \left(\frac{z}{w}\right)^2 = \frac{z}{w}$

$\frac{z}{w} = l$

$z = wl$

$\frac{dz}{dw} = w\frac{dl}{dw} + \frac{dw}{dw} l$

$\boxed{\frac{dz}{dw} = l + w\frac{dl}{w}}$

$l + w\frac{dl}{dw} + l^2 = l$

$w\,dl + l^2 dw = 0$

$w\,dl = -l^2 dw$

$\frac{-1}{l^2} dl = \frac{1}{w} dw$

(8) The I.V.P $y' = ty(1 + \tan y)$, $y(0) = 0$

(a) has exactly one solution.

(b) has many solutions.

(c) has no solution.

(d) cannot tell.

$\frac{\sin}{\cos}$

$y' = 0$

$(0, 0)$

(9) The integral curves (solution curves) of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ are

(a) circles

(b) lines

(c) ellipses

(d) parabolas

$y\,dy = -x\,dx$

$y^2 = -x^2 + c$

$\boxed{y^2 + x^2 = c}$

$\underline{\text{circles}}$

(10) The general solution of the differential equation $xy' - y = 2x\ln x$, $x > 0$ is

(a) $y(x) = x\ln x + cx$

(b) $y(x) = x^{-1}(\ln x + c)$

(c) $y(x) = x(\ln^2 x + c)$

(d) $y(x) = \ln^2 x + cx$

$y' - \frac{1}{x}y = 2\ln x$

$p(t) = \frac{1}{x}$ $g(t) = 2\ln x$

$\mu(t) = e^{-\ln x} = \frac{1}{x}$

$y(x) = x\left[\int \frac{2\ln x}{x}dx + c\right]$

$\ln x = u$

$du = \frac{1}{x}dx$

$2u\,du$

$y(x) = \frac{1}{x}\left[u^2 + c\right]$

$y(x) = x(\ln x)^2 + xc$

(11) The general solution of the differential equation $2y'' + 2y' + y = 0$ is

(a) $y(t) = c_1e^t\sin t + c_2e^t\cos t$

(b) $y(t) = c_1e^{-t}\sin t + c_2e^{-t}\cos t$

(c) $y(t) = c_1e^{2t}\sin t + c_2e^{2t}\cos t$

(d) $y(t) = c_1e^{-t}\sin(2t) + c_2e^{-t}\cos(2t)$

$2r^2 + 2r + 1 = 0$

$r^2 + r + \frac{1}{2} = 0$

$\frac{-1 \pm i}{2}$

$\boxed{-\frac{1}{2} \pm \frac{1}{2}i}$

(12) The I.V.P. $\frac{dy}{dx} = xy^{1/2}$, $y(0) = 0$

(a) has no solution.

(b) has one solution.

(c) has at least two solutions.

(d) none of the above.

$\frac{dy}{\sqrt{y}} = x\,dx$

$2\sqrt{y} = \frac{x^2}{2}$

(13) A drug is infused into a patient's bloodstream at a constant rate of r grams per second. Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at time t. A differential equation for the amount $x(t)$ is

$\frac{dx}{dt} = r - k\,x(t)$

(a) $x'(t) = x(t) - r$

(b) $x'(t) = r - kx(t)$, for some constant k

(c) $x'(t) = rx(t) - k$, for some constant k

(d) $x'(t) = x(t) - rx(t)$

$2x^6$ 4

$10x$

(14) An integrating factor for the differential equation $xy\,dx + (2x^2 + 3y^2 - 1)dy = 0$ is

(a) $\frac{3}{y}$

(b) y^3

(c) y^{-3}

(d) x^3

xy^4 3 $4x$ $2x^2y^{-3}$

$4xy^{-3}$

$\frac{3x}{xy} = \frac{3}{y}$

xy^{-2}

$\boxed{-2xy^{-3}}$ 3

$\frac{3x}{2x^2+3}$

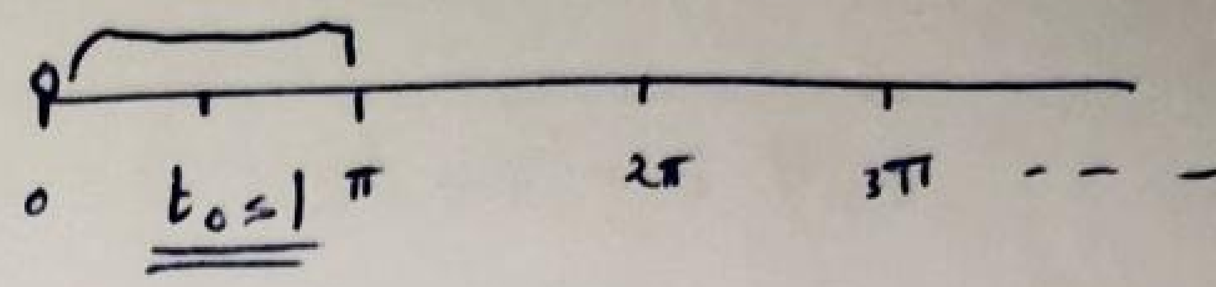
$3x\,dx + \left(\frac{6x^2}{y}\right.$

(15) The longest interval in which a solution of the I.V.P. $(1-\ln t)y' + ty = \cot t$, $y(1) = 1$ is certain to exist is

(a) $(0, \pi)$
(b) $(1, \pi)$
(c) $(0, \infty)$
(d) $(0, e)$

$\boxed{t > 0}$

$(0, \pi)$



(16) If $y(t) = c_1 e^{2t} + c_2 e^{-3t}$ is the general solution of the diff. equation $y'' + ay' + by = 0$

(a) $a = 3, b = 2$
(b) $a = -3, b = 2$
(c) $a = 1, b = -6$
(d) $a = -1, b = 6$

$(r-2)(r+3) = 0$

$r^2 - 2r + 3r - 6$

$\boxed{a = 1}$ $\boxed{b = -6}$

(17) The solution of the I.V.P. $y' = -\frac{t}{y}$, $y(0) = y_0$ is defined on the interval

$\frac{dy}{dt} = \frac{-t}{y}$

(a) (y_0, ∞) if $y_0 < 0$
(b) $(-\infty, y_0)$ if $y_0 > 0$
(c) $(-|y_0|, |y_0|)$
(d) $(-\infty, \infty)$

$y\,dy = t\,dt$ $\boxed{y_0 \neq 0}$

$y^2 = -t^2 + c \Rightarrow \boxed{y^2 + t^2 = c}$

$y_0^2 + 0 = c$

$y = \pm\sqrt{2y_0^2 - t^2}$

$y_0 = \sqrt{2c}$

$y_0^2 = 2c$

$\boxed{c = \frac{y_0^2}{2}}$

(18) Let $y(t)$ be the solution of the I.V.P. $y' + y = t$, $y(0) = 4$ then $y(1) =$

$p(t) = 1$

(a) $5e^{-1}$
(b) $5e$
(c) 5
(d) $4e^{-1}$

$y^2 + x^2 = 4$

$x = 3$

$y^2 + 9 = 4$

$p(t) = 1$ $g(t) = t$

$\mu(t) = e^t$

$y(t) = (t-1) + \frac{5}{e^t}$

$te^t - e^t$

(19) An integrating factor for the differential equation $(x^2-9)y' + xy = 1$, $x > 3$, is

(a) $\ln(x^2-9)$
(b) $e^{(x^2-9)}$
(c) $\ln\sqrt{x^2-9}$
(d) $\sqrt{x^2-9}$

$\frac{2x}{x^2-9}$

$u = x^2 - 9$

$\frac{2x - x}{x^2-9}$

$\boxed{\frac{1}{y}}$

$y = -1 + c$

$e^{\int \frac{x}{x^2-9}\,dt}$

$\frac{x}{x^2-9}$

$e^{\frac{1}{2}\ln(x^2-9)}$

$\sqrt{x^2-9}$

$M = xy - 1$

$N = x^2 - 9$

$M_y = x$

$N_x = 2x$

$\frac{M_y - N_x}{M} = f(x)$

$(x^2-9)^{\frac{1}{2}} xy$

(20) If $y_1(t)$ and $y_2(t)$ are solutions of $y'' + e^t y' + (\sin t)y = \cos t$ then $y_1(t) + y_2(t)$ is a solution

(a) True.

(b) False.

Question 2 (8 points) Solve the initial value problem $y' + \frac{1}{t}y = y^2$, $y(1) = 1$. Find the interval in which the solution is defined.

$$y^{-2}y' + \frac{1}{t}y^{-1} = 1$$

$$y^{-1} = v$$

$$y^{-2}y' = -v'$$

$$-v' + \frac{1}{t}v = 1$$

$$v' - \frac{1}{t}v = -1$$

$$p(t) = \frac{-1}{t}, \quad g(t) = -1$$

$$\mu(t) = \frac{1}{t}$$

$$v(t) = t\left[\int \frac{-1}{t}\,dt + c\right]$$

$$v(t) = t[-\ln t + c]$$

$$\frac{1}{y(t)} = -t\ln t + ct \qquad (y(1) = 1)$$

$$\frac{1}{1} = -(1)\ln(1) + c(1)$$

$$\boxed{c = 1}$$

$$\frac{1}{y(t)} = -t\ln t + t$$

$$y(t) = \frac{1}{t - t\ln t} = \frac{1}{t(1-\ln t)}$$

Interval :-

$t > 0$, $t \neq e$, $\ln t \neq 1$

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Question (12 points) Solve the I.V.P. $y'' = (y')^2 t$, $y'(0) = 2$, $y(0) = 1$.

$y' = v$

$y'' = v'$

$v' = v^2 t$

$\frac{dv}{v^2} = t\,dt$

$\frac{-1}{v} = \frac{t^2}{2} + c \qquad \left(\begin{array}{l} v(0) = 2 \\ \frac{-1}{2} = 0 + c \Rightarrow c = \frac{-1}{2} \end{array} \right)$

$\cancel{v(t) = \frac{-2}{t^2}} \qquad \cancel{v(0) = 2}$

$\cancel{y = \int \frac{-2}{t^2}\,dt = \frac{2}{t} + c}$

$\frac{-1}{v} = \frac{t^2}{2} - \frac{1}{2}$

$v = \frac{-2}{t^2-1} \Rightarrow y = \int \frac{-2}{t^2-1}\,dt$

$-2 = (t-1)A + (t+1)B$

$t = 1 \Rightarrow 2B = -2 \Rightarrow \boxed{B = -1}$

$t = -1 \Rightarrow -2A = -2 \Rightarrow \boxed{A = 1}$

$y = \int \frac{1}{t+1} + \frac{-1}{t-1}\,dt$

$y(t) = \ln|t+1| - \ln|t-1| + c \qquad y(0) = 1$

$1 = \ln 1 - \ln 1 + c$

$c = 1$

$\boxed{y(t) = \ln\left|\frac{t+1}{t-1}\right| + 1}$ solution.

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