Question One (26 points) Circle the most correct answer:

1. The differential equation $\frac{dy}{dx} - y^2 + y = 0$ is

   (a) not exact

   (b) separable

   (c) Bernoulli

   (d) All of the above

$y' - y^2 + y = 0$ $\quad$ 6 $\quad$ $y' + y = y^2$

$\frac{dy}{dx} = y^2 - y$

$\int \frac{dy}{y^2 - y} = \int dx$

2. The DE $2xy\, dx + (x^2 - y^2)\, dy = 0$ is

   (a) exact

   (b) linear

   (c) separable

   (d) exact and separable

$M = 2xy$ $\quad$ $M_y = 2x$

$N = x^2 - y^2$ $\quad$ $N_x = 2x$

exact

3. Suppose the change on a temperature of a hot cup obeys to Newton's law of cooling. The hot cup initially has a temperature of $100F$ and brought to a room with temperature $30F$. If the temperature of the cup becomes $65F$ in $\ln 2$ minutes, then the temperature of the cup after $\ln 7$ minutes is

   (a) $50F$

   (b) $20F$

   (c) $40F$

   (d) $30F$

$T_0 = 100$

$T_m = 30$

$T(\ln 2) = 65$

$T(\ln 7) = 30 + 70e^{-\ln 7}$

$30 + 10$

$30 + 9 = 40$

$\frac{dT}{dt} = K(T - T_m)$

$\frac{dT}{dt} = KT - KT_m$

$k \ln 2$ $\quad$ $a = K$

$$\frac{x\acute{y}}{x} - \frac{y \ln \frac{y}{x}}{x} = \frac{5x}{x} \qquad v = \frac{y}{x}$$

$$\acute{y} - v \ln v = 5$$

8

4. The integrating factor of the DE $(x^2 - xy)y' = y^2 - 3xy, x > 0$ is nonlinear

(a) y

(b) x

(c) xy

(d) None of the above

$M = 3xy - y^2 \qquad M_y = 3x - 2y$

$N = x^2 - xy \qquad N_x = 2x - y$

$\frac{x - y}{x(x-y)} = \frac{1}{x} \qquad e^{\int \frac{1}{x} dx} \qquad 3x - 2y - 2x + y$

$= e^{\ln x} = \boxed{x}$

5. The DE $ty' - y \ln \frac{y}{t} = 5t, t > 0$ is

(a) 1^{st} order, linear, homogeneous ODE X

(b) 1^{st} order, linear, non-homogeneous ODE X

(c) 1^{st} order, non-linear, non-homogeneous ODE

(d) 1^{st} order, non-linear, homogeneous ODE ✓

First / non linear / ODE

$v = \frac{y}{t} \qquad \acute{y} - \frac{\ln v}{\frac{t}{t}} = \frac{5t}{t}$

6. The IVP: $\frac{dx}{dt} = \sqrt{x}, x(0) = 1$

(a) has more than two solution

(b) has a unique solution

(c) has no solution

(d) has two solutions

$\acute{x} = \sqrt{x}$ non linear

$\acute{x} = \frac{1}{2\sqrt{x}} \qquad x \neq 0 \qquad \therefore$ unique sol

7. If $y' - 2y = e^t$, $y(0) = 2$, then $y(\ln 2) =$

(a) 0

(b) 2

(c) 5

(d) 10

$p(t) = -2 \qquad g(t) = e^t$

$\mu(t) = e^{\int -2\, dt} = \boxed{e^{-2t}}$

$e^{2t}\left[\int e^{-2t} \cdot e^t\, dt + c\right]$

$\int e^{-t}\, dt$

$e^{2t}\left[-e^{-t} + c\right]$

$y(t) = -e^t + ce^{2t}$

$y(0) = 1 + c = 2$

$\boxed{c = 1}$

$y(t) = -e^t + e^{2t}$

$y(\ln 2) = -e^{\ln 2} + e^{\ln 4}$

$-2 + 4 =$

2

8. The half-life time of a radioactive material is $\ln 8$ years. After $\ln 64$ years, this material becomes 10 gm. The initial amount of this material is

(a) 32 gm

(b) 40 gm

(c) 24 gm

(d) 12 gm

general $y(t) = y_0 e^{kt}$

$y(\ln 8) = \frac{1}{2} y_0 \qquad y(\ln 64) = 10\,gm$

$\frac{-\ln 2}{\ln 8} \qquad = 2\ln\frac{1}{2} = \ln 1 - \ln 4$

$\frac{\ln \frac{1}{2}}{\ln 8} \cdot 2\ln 8$

$\frac{1}{2} y_0 = y_0 e^{k \ln 8}$

$\ln \frac{1}{2} = \ln e^{k \ln 8}$

$-\ln 2 = k \ln 8$

$\frac{1}{2} y_0 = e^{k \ln 8}$

$\frac{1}{2} = e^{k \ln 8}$

$10 = y_0 e^{k \ln 64}$

$10 = y_0 e^{-\ln 4}$

$4 \times 10 = y_0 \frac{1}{4}$

10

9. If $y(x)$ solves the IVP: $y' - y^2 + y = 0,\ y(0) = -1$, then, $\lim_{x\to\infty} y(x) =$

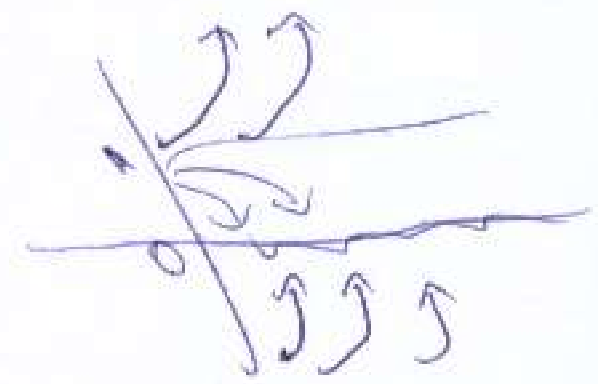
(a) 0
(b) -1
(c) 1
(d) $-\infty$

$y = -1$

$y' + y = y^2 - y$

$y' = y^2 - y$

$y(y-1) = 0, 1$



10. The behavior of solution diverges in the DE $a > 0$

(a) $y' = 1 + y$ $a = 1$
(b) $y' = 1 - y$ $a = -1$
(c) $y' + 2y = -2$ $a = -2$
(d) $y' + 3 = -y$ $a = -1$

11. The largest interval in which the solution of the IVP $(\ln t)y' + \frac{1}{(t-3)}y = t^2,\ y(2) = 4$ is defined

linear

(a) $(1,3)$
(b) $(3,\infty)$
(c) $(0,3)$
(d) $(0,\infty)$

$y' + \frac{1}{(\ln t)(t-3)} = \frac{t^2}{\ln t}$

$t \neq 3$ $t \neq 1$

$y' = t^3 y + y^2 = (1)(2) + (4) = 6$

$y(1) = 2$

12. If $y(t)$ solves the DE $y' - t^3 y = y^2$, then the slope of $y(t)$ at the point $(1,2)$ is

$p(t) = -t^3$ $n = 2$

(a) 2
(b) 4
(c) 6
(d) 0

13. The solution of the IVP $y' - 3y - 9 = 0,\ y(0) = 1$ is

$v = y^{-1}$

(a) $y(t) = e^{3t}$
(b) $y(t) = 2e^{3t} - 1$
(c) $y(t) = 3e^{3t} - 2$
(d) $y(t) = 4e^{3t} - 3$

$\frac{1}{y} = v$

$y = \frac{1}{v}$

$y' = 3y + 9$ $a = 3$, $b = -9$ $\frac{b}{a} = -3$

$y(t) = \frac{1}{v}$

$y(t) = -3 + [1 + 3]e^{3t}$
$= -3 + 4e^{3t}$

$$2y - x - 2x$$
$$\frac{2y-3x}{y(y-x)}$$

Question Two (7 points) Find an explicit solution for the IVP:

$$\frac{dy}{dx} = \frac{y(x-y)}{x^2}, \qquad y(1) = -1, \qquad x > 0$$

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$$v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{v(1-v)}{1^2}$$

$$y = vx \qquad y' = \frac{dy}{dx} = v + xv'$$

$$xv' = \cancel{v} - v^2 - \cancel{v} \qquad xv' = -v^2$$

$$x\frac{dv}{dx} = -v^2\,dx \qquad \int -v^{-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\left(\frac{-1}{v}\right) = \ln x + c \Rightarrow \frac{1}{v} = \ln x + c$$

$$v = \frac{1}{\ln x + c} \Rightarrow \frac{y}{x} = \frac{1}{\ln x + c}$$

$$\boxed{y = \frac{x}{\ln x + c}} \qquad y(1) = -1 = \frac{1}{0 + c}$$

$$-1 = \frac{1}{c}$$

$$\boxed{c = -1}$$

$$\therefore \boxed{y(x) = \frac{x}{\ln x - 1}}$$

Question Three (8 points) Solve the IVP:

$$xy^3 + (x^2y^2+1)y' = 0, \qquad y(2) = 1, \qquad x > 0, \qquad y > 0$$

8

$M = xy^3 \qquad M_y = 3xy^2$ not exact

$N = x^2y^2+1 \qquad N_x = 2y^2x$

$\frac{M_y - N_x}{M} = \frac{3xy^2 - 2xy^2}{xy^3} = \boxed{\frac{1}{y}}$

$e^{-\int \frac{1}{y}} = e^{-\ln y} = \boxed{\frac{1}{y}}$

Now, $M = xy^2 \qquad M_y = 2xy$

$N = x^2y + \frac{1}{y} \qquad N_x = 2xy$ } exact ✓

$\therefore \quad \psi_x = M \qquad \psi_y = N$

$\psi = \int \psi_x \, dx = \int M \, dx = \int xy^2 \, dx = \frac{x^2y^2}{2} + g(y)$

To find $g(y) \Rightarrow \psi_y = N \Rightarrow \frac{1}{2}(2yx^2) + g'(y) = x^2y + \frac{1}{y}$

$\cancel{x^2y} + g'(y) = \cancel{x^2y} + \frac{1}{y} \qquad \int g'(y) = \int \frac{1}{y} \Rightarrow \boxed{g(y) = \ln y}$

$\therefore \quad \psi = \frac{1}{2}x^2y^2 + \ln y = c$

$(\frac{1}{2})(4)(1) + \cancel{\ln 1} = c \qquad \boxed{c = 2}$

$$\boxed{\frac{1}{2}x^2y^2 + \ln y = 2}$$

Question Four (9 points) Given the IVP:

$$y' + \frac{4}{x}y = x^3y^2, \qquad y(1) = \frac{1}{2}, \qquad x > 0$$

(1) Find the unique solution

(2) Find the interval where the solution is defined

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nonlinear, solve

$p(x) = \frac{4}{x}$ $\quad g(x) = x^3$ $\quad n = 2$, $\boxed{1-n = -1}$ $\quad v = y^{1-n}$, $v = y^{-1}$, $v = \frac{1}{y}$, $\boxed{y = \frac{1}{v}}$

$v' + (1-n)p(x)\, v = (1-n)g(x)$

$B^* \Rightarrow v' + \frac{-4}{x} v = -x^3$

$p(t) = \frac{-4}{x}$ $\quad g(t) = -x^3$

$\mu = e^{\int -\frac{4}{x}} = e^{-4\ln x} = \boxed{x^{-4}}$

$x^4\left[\int x^{-4}\cdot -x^3 \, dx + c\right] = x^4\left[-\int \frac{1}{x}\, dx + c\right]$

$x^4\left[-\ln x + c\right] = (x^4)(-\ln x) + x^4 c = v(t)$

$y = \frac{1}{(x^4)(-\ln x) + x^4 c} \Rightarrow \frac{1}{2} = \frac{1}{0 + c}$

$\frac{1}{2} = \frac{1}{c}$ $\quad \boxed{c = 2}$

~~$x^4[2 - \ln x]$~~

$$\boxed{y(t) = \frac{1}{(x^4)(-\ln x) + 2x^4}}$$

unique ✓

$$\boxed{y(t) = \frac{1}{x^4(2 - \ln x)}}$$

interval: $x \neq 0$, $x \neq 1$

$\ln x \neq 2$

$e^{\ln x} = e^2$

Interval $(0, e^2)$

4 in   2 out

$\boxed{80+2t}$

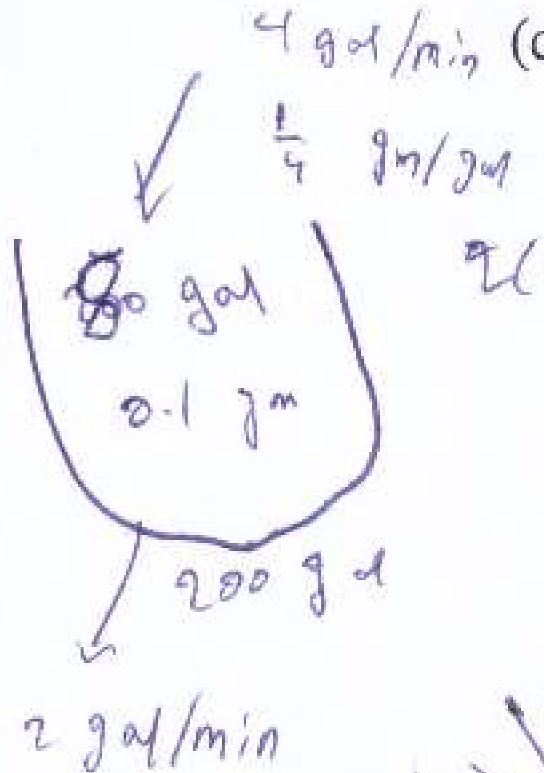
10

Question Five (10 points) A tank of capacity 200 gal has initially 0.1 gm of toxic wastes dissolved in 80 gal of water. Water with toxic wastes starts flow into the tank at rate 4 gal/min and flow out at rate 2 gal/min. The incoming water contains $\frac{1}{4}$ gm/gal of toxic wastes.

(a) Write IVP that models the change of toxic wastes in the tank over time.

(b) Find the amount of toxic wastes in the tank at any time.

(c) Find the amount of toxic wastes in the tank when it becomes to overflow.



$q(0) = 0.1$

$\frac{dq}{dt} = (\text{rate in})(\text{conc in}) - (\text{rate out})(\text{conc out})$

$\frac{dq}{dt} = 4 \times \frac{1}{4} - \frac{2\,q}{80+2t}$

(a) $\boxed{\frac{dq}{dt} + \frac{2q}{80+2t} = 1}$ $p(t) = \frac{2}{80+2t}$ $g(t) = 1$

$\mu = e^{\int \frac{2}{80+2t}dt} = e^{\int \frac{1}{u}du} = e^{\ln(80+2t)}$ $u = 80+2t$ $\frac{du}{dt} = 2$ $dt = \frac{du}{2}$

$\cancel{e^{\ln 80 + 2t}}$ $= \cancel{\boxed{80\,e^{2t}}}$

$= (80+2t)$

(b) $\frac{1}{80+2t}\left[\int (80+2t)\,dt + c\right] \Rightarrow q(t) = \frac{(80+2t)^2}{\cancel{(80+2t)}\,2} + \frac{c}{80+2t}$

$q(t) = \frac{(80+2t)}{2} + \frac{c}{80+2t}$ $\Rightarrow$ $\cancel{q(0) = 80 + \frac{c}{80} = \frac{1}{100} - \frac{80}{100}}$ $\times 100$

$\cancel{\frac{c}{80} = \frac{(-7999)(80)}{100}}$ $\cancel{\boxed{c = \frac{(-7999)(8)}{10}}}$ $1 - 8000$

$q(0) = 0.1 = \frac{1}{100}$

$\cancel{q(t) = 80+2t + \frac{\frac{(-7999)(8)}{10}}{80+2t}}$

Good Luck

$q(0) = 40 + \frac{c}{80} = \frac{1}{100} - \frac{4000}{100}$ $\boxed{c = \frac{-3999(80)}{100}}$

(d) overflow every one minute it rises 2 galons

1 min → 2 gal
x min → 120 gal
$2x = 120$
$\boxed{x = 60 \text{ min}}$

$q(60) = 200 + \frac{(-3999)(0.8)}{200}$ min