$t \geq 0$, show that

$\frac{48}{50}$

Birzeit University-Mathematics Department
Math 331-Ordinary differential Equations

Second Exam
Name: [redacted] ............ Section: (10:00), (12:00)

Summer 2011/2012
Number: [redacted]

**Question 1**(24 points) Circle the correct answer

1. The general solution of the differential equation $y''' + 2y'' + 2y' = 0$ is

   (a) $y = c_1 + c_2 \cos t + c_3 \sin t$.

   (b) $y = c_1 + c_2 e^t \cos t + c_3 e^t \sin t$.

   (c) $y = c_1 t + c_2 e^{-t} \cos t + c_3 e^{-t} \sin t$.

   (d) $y = c_1 + c_2 e^{-t} \cos t + c_3 e^{-t} \sin t$.

$r^3 + 2r^2 + 2r = 0$

$r(r^2 + 2r + 2) = 0$

$r = 0$,

$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$

2. The form of a particular solution of the differential equation $y''' + y'' = t + e^{-t}$

   (a) $Y = At + B + Ce^{-t}$.

   (b) $Y = t(At + B) + Ce^{-t}$.

   (c) $Y = t^2(At + B) + Cte^{-t}$.

   (d) $Y = t^2(At + B) + Ce^{-t}$.

$\boxed{Y_p = t(At + B)e^{-t}}$

$r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r = 0,\ r = -1$

$y_c = c_1 + c_2 t + c_3 e^{-t}$

3. The general solution of the differential equation $t^2 y'' + y = 0$ is

   (a) $y = c_1 \cos t + c_2 \sin t$.

   (b) $y = c_1 t^{1/2} \cos(\frac{\sqrt{3}}{2} \ln t) + c_2 t^{1/2} \sin(\frac{\sqrt{3}}{2} \ln t)$.

   (c) $y = c_1 t^{1/2} \cos(\sqrt{3}t) + c_2 t^{1/2} \sin(\sqrt{3}t)$.

   (d) $y = c_1 \cos(\sqrt{3} \ln t) + c_2 \sin(\sqrt{3} \ln t)$.

$r(r-1) + 1 = 0$

$r^2 - r + 1 = 0$

$r = \frac{+1 \pm \sqrt{1 - 4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\lambda = \frac{1}{2}$

$\mu = \frac{\sqrt{3}}{2}$

$y = c_1 t^{1/2} \cos(\frac{\sqrt{3}}{2}) \ln t + c_2 t^{1/2} \sin(\frac{\sqrt{3}}{2}) \ln t$.

4. A lower bound for the radius of convergence of the series solutions at $x = 1$ of the differential equation $(x^2 + 1)y'' + \frac{x}{x+1}y' + y = 0$ is

   (a) $\rho = 1$.

   (b) $\rho = 1/2$.

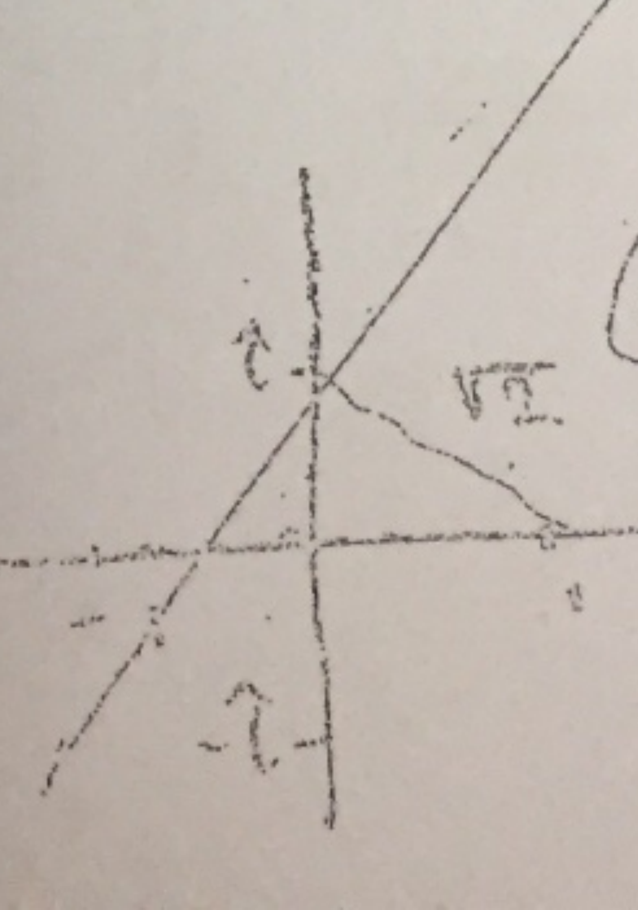
   (c) $\rho = \sqrt{2}$.

   (d) $\rho = 2$.

$(x+1)(x^2 + 1) = 0$

$x = -1$

$x = \pm i$



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5. The point $x = 1$ for the equation $(x^2-1)y'' + \frac{x}{x-1}y' + y = 0$ is

$P_0 = \lim \frac{x}{(x-1)(x^2-1)}(x-1)^2 \to \infty$ irr.

(a) Ordinary point.

(b) Regular singular point.

(c) Irregular singular point.

(d) None.

6. Consider the I.V.P. $y'' + e^x y = 0$, $y(0) = 1$, $y'(0) = 1$. The coefficient of $x^3$ in a series solution of this I.V.P. at $x = 0$, $a_3 =$ $a_0 = 1$, $a_1 = 1$

(a) $-2$.

(b) $-\frac{1}{3}$.

(c) $-\frac{2}{3}$.

(d) $2$.

$\phi''(0) = -e^x\phi(0) = -1(1) = -1$

$\phi'''(0) = -e^x\phi'(0) + -1e^x\phi(0)$

$= -1(1) + -1(1) = -2$

$a_3 = \frac{\phi'''}{3!} = \frac{-2}{6} = \frac{-1}{3}$

7. Let $F(s) = \mathcal{L}\{f(t)\}$. Then $\mathcal{L}\{f(ct)\} =$

$\int_0^\infty e^{-st} f(ct)\, dt$ $\quad ct = z$, $dz = c\,dt$

$\frac{1}{c}\int_0^\infty e^{-(\frac{s}{c})z} f(z)\, dz$

$= \frac{1}{c}F\left(\frac{s}{c}\right)$

(a) $\frac{1}{c}F(s)$.

(b) $F\left(\frac{s}{c}\right)$.

(c) $\frac{1}{c}F\left(\frac{s}{c}\right)$.

(d) $cF(s)$.

8. Let $g(t) = \int_0^t \tau^n e^\tau d\tau$. Then, $G(s) =$

(a) $\frac{n!}{(s-1)^n}$.

(b) $\frac{n!}{s(s-1)^{n+1}}$

(c) $\frac{n!}{(s-1)^{n+1}}$

(d) $\frac{(n+1)!}{(s-1)^{n+1}}$

$f(t) = \tau^n e^\tau \Rightarrow F(s) = \frac{n!}{(s-1)^{n+1}}$

$G(s) = \frac{F(s)}{s} = \frac{n!}{s(s-1)^{n+1}}$



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9. The general solution of the differential equation $(t-1)^2y'' - (t-1)y' + y = 0$, $t > 1$ is

(a) $y = c_1(t-1) + c_2(t-1)\ln(t-1)$.

(b) $y = c_1(t-1) + c_2 t(t-1)$.

(c) $y = c_1(t-1)^{-1} + c_2(t-1)^{-1}\ln(t-1)$.

(d) $y = c_1(t-1) + c_2(t-1)^{-1}$.

$r(r-1) - r + 1 = 0$

$r^2 - r - r + 1 = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$\Rightarrow y = c_1(t-1) + c_2(t-1)\ln(t-1)$

10. Let $F(s) = \frac{s}{s^2+2s+3}$. Then, $f(t) =$

(a) $e^{-t}\cos(\sqrt{2}t)$.

(b) $e^{-t}\cos(\sqrt{2}t) - \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$.

(c) $e^{-t}\sin(\sqrt{2}t) + \frac{1}{\sqrt{2}}\cos(\sqrt{2}t)$.

(d) $e^{-t}\cos(\sqrt{2}t) + \sin(\sqrt{2}t)$.

$F(s) = \frac{s}{(s+1)^2 + (\sqrt{2})^2}$

$= \frac{(s+1)}{(s+1)^2+(\sqrt{2})^2} - \frac{1}{(s+1)^2+(\sqrt{2})^2}$

$= e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t$

11. The Laplace transform of the function

$$f(t) = \begin{cases} t, & t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

(a) $\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$.

(b) $\frac{1}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$.

(c) $\frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(d) $\frac{1}{s} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$.

$t(1 - U_1(t)) + (U_1(t) - U_2(t))$

$t - tU_1(t) + U_1(t) - U_2(t)$

$t - (t-1)U_1(t) - U_1(t) + U_1(t) - U_2(t)$

$\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$

12. The solution of the initial value problem $y'' - y = 1$, $y(0) = 0$, $y'(0) = 0$.

(a) $\cosh(t)\,u_1(t)$.

(b) $\sinh(t)\,u_1(t)$.

(c) $-1 + \cosh t$.

(d) $-1 + \sinh t$.

$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s}$

$Y(s) = \frac{1}{s(s^2-1)}$

$Y(s) = \frac{1}{s(s^2-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$

$A = -1$ $B = \frac{1}{2}$, $C = \frac{1}{2}$

$y(t) = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$

$= -1 + \frac{e^t + e^{-t}}{2} = -1 + \cosh t$

**Question 2**(8 points) Find the general solution of $y'' + y = \cot(t)$, $0 < t < \pi$.

$p(t) = 1$ $\quad y_c \Rightarrow r^2 + 1 = 0$

$g(t) = \cot(t)$ $\quad r = \pm i$

so: $y_c = c_1 \underbrace{\cos t}_{y_1} + c_2 \underbrace{\sin t}_{y_2}$.

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & +\cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$u_1 = -\int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W}\, ds = -\int_{t_0}^{t} \cancel{\sin s} \left(\frac{\cos s}{\cancel{\sin s}}\right) ds$$

$$= -\int_{t_0}^{t} \cos s \, ds = -\sin t$$

$$u_2 = \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W}\, ds = \int_{t_0}^{t} \cos s \cdot \frac{\cos s}{\sin s}\, ds$$

$$= \int_{t_0}^{t} \frac{1 - \sin^2 s}{\sin s}\, ds = \int_{t_0}^{t} \csc s \, ds - \underbrace{\int_{t_0}^{t} \sin s \, ds}_{\to \cos t}$$

$$\int_{t_0}^{t} \csc s \, ds \Rightarrow \int_{t_0}^{t} \frac{\csc s(\csc s + \cot s)}{(\csc s + \cot s)}\, ds = \int_{t_0}^{t} \frac{\csc^2 s + \cot s \csc s}{\csc s + \cot s}\, d$$

let $z = \cot s + \csc s$

$dz = -1(\cot s \csc s + \csc^2 s)$

So: $\int_{t_0}^{t} \csc s \, ds = -\ln|\csc t + \cot t|$

So: $\Rightarrow y(t) = u_1 y_1 + u_2 y_2$

$= -\cos t \sin t + \sin t \left[\cos t - \ln|\csc t + \cot t|\right]$



$\Rightarrow$ general solu.

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Question 3(10 points) Find the first three terms in $y_1(x)$ and $y_2(x)$ in series solutions of the differential equation $y'' - xy' - y = 0$ near $x = 0$ $\Rightarrow$ ordinary point

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - xy' - y \Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - (n+1) a_n] x^n = 0$$

$$2a_2 = a_0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$(n+2)(n+1) a_{n+2} = (n+1) a_n$$

$$a_{n+2} = \frac{a_n}{n+2}, \quad n > 1$$

$$a_3 = \frac{a_1}{3}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{(2)(4)}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{(3)(5)}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{(2)(4)(6)} \ldots$$

So: $y(x) = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{8} x^4 + \frac{a_1}{15} x^5 \ldots$

$$y_1(x) = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right)$$

$$y_2(x) = a_1\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \ldots\right)$$

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$$y(x) = \sum_{n=1}^{\infty} (a_{n+2})(n+2) x^n$$

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**Question 4**(8 points) Use Laplace transform to solve the I.V.P.

$$y'' + y = tu_1(t), \quad y(0) = 1, y'(0) = 1$$

$$tu_1(t) = ((t-1)+1)\,u_1(t) = (t-1)u_1(t) + u_1(t)$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

$$Y(s)(s^2+1) - 1(s+1) = \frac{e^{-s} + se^{-s}}{s^2}$$

$$Y(s) = \frac{e^{-s} + se^{-s}}{s^2(s^2+1)} + \frac{s+1}{s^2+1}$$

let $F(s) = \dfrac{1}{s^2(s^2+1)}$

$$\frac{1}{s^2(s^2+1)} = \frac{As+C}{s^2} + \frac{Ds+E}{s^2+1}$$

$$1 = \cancel{As^3} + \cancel{As} + \cancel{Cs^2} + \cancel{C} + \cancel{Ds^3} + Es^2$$

$A + D = 0$ , $C + E = 0$ , $C = 1$, $A = 0$, $E = -1$, $D = 0$

So: $F(s) = \dfrac{1}{s^2} + \dfrac{-1}{s^2+1}$

$f(t) = t - \sin t$

let $H(s) = \dfrac{1}{s(s^2+1)} = \dfrac{1}{s} + \dfrac{As+D}{s^2+1}$

$$\cancel{1} = s^2 + \cancel{1} + As^2 + Ds$$

$D = 0$ , $A + 1 = 0 \Rightarrow A = -1$

So: $H(s) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$m(s) = 1 - \cos t$

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let $N(s) = \dfrac{s+1}{s^2+1} = \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$

so: $n(s) = \cos t + \sin t$.

Then $\Rightarrow$ $Y(s) = F(s)e^{-s} + M(s)e^{-s} + N(s)$

$y(t) = U_1(t)\,f(t-1) + U_1(t)\,m(t-1) + n(t)$

$y(t) = U_1(t)\left[(t-1) - \sin(t-1)\right] + U_1(t)\left[1 - \cos(t-1)\right] + \cos t + \sin t$