Key Solution

BIRZEIT UNIVERSITY

Department of Mathematics

Second Exam

Differential Equations (Math 331) **July 29, 2018**

Instructor: Dr. Ala Talahmeh **Time: Two Hours** **Summer 2018**

Name: ______________ **Number:** __________ **Section:** __________

Question one (36%). Circle the correct answer.

1. Let $y(t)$ be the solution of the IVP:

$$y'' - 2y' + y = 0, \ y(0) = y'(0) = 1.$$

Then $y(\ln 2) =$

(a) 2

(b) $2\ln 2$

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

Aux. eq. $r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$

$\therefore y_h = c_1 e^t + c_2 t e^t$

$y(0) = \boxed{c_1 = 1}$, $y' = c_1 e^t + c_2 e^t + c_2 t e^t$

$y'(0) = c_1 + c_2 = 1 \Rightarrow c_2 = 0$

$\therefore y = e^t$, then $y(\ln 2) = e^{\ln 2} = 2$

2. Suppose that y_1 and y_2 are solutions of the **nonhomogeneous** differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t),$$

and y_3 and y_4 are solutions of the corresponding **homogeneous** differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

One of the following statements is **false**.

(a) $5y_3$ is a solution of the homogeneous differential equation

(b) $y_3 - y_1$ is a solution of the nonhomogeneous differential equation

(c) $y_3 + y_4$ is a solution of the homogeneous differential equation

(d) $y_1 - y_3$ is a solution of the nonhomogeneous differential equation

Given that $L[y_1] = L[y_2] = g(t)$ and $L[y_3] = L[y_4] = 0$

Now, $L[y_3 - y_1] = L[y_3] - L[y_1] = 0 - g(t) = -g(t) \neq g(t)$.

$L[5y_3] = 5L[y_3] = 0$ ✓

$L[y_3 + y_4] = L[y_3] + L[y_4] = 0$ ✓

$L[y_1 - y_3] = L[y_1] - L[y_3] = g(t) - 0 = g(t)$ ✓

7. If $a < b < c < d$ are the roots of the auxiliary equation of the ODE

$$y^{(4)} + 3y^{(3)} - 4y'' - 12y' = 0,$$

then $a + 2b + c + d$ is equal to

(a) -5

(b) -1

(c) 2

(d) 5

Aux. eq. $r^4 + 3r^3 - 4r^2 - 12r = 0$

$\Rightarrow r^3(r+3) - 4r(r+3) = 0$

$\Rightarrow (r^3 - 4r)(r+3) = 0$

$\Rightarrow r(r-2)(r+2)(r+3) = 0$

$\Rightarrow r = 0, 2, -2, -3$

$\Rightarrow \underset{a}{-3} < \underset{b}{-2} < \underset{c}{0} < \underset{d}{2}$

8. Consider the following ODE

$$y'' + 9y' = t^2\cos(3t).$$

How many terms are in the correct expression for the form of y_p?

(a) 2

(b) 4

(c) 6

(d) 8

$r^2 + 9r = 0 \Rightarrow r = 0, -9$

$\therefore y_h = c_1 + c_2 e^{-9t}$.

The form of $y_p = (At^2 + Bt + C)\cos(3t) + (Dt^2 + Et + F)\sin(3t)$.

9. Which one of the following is equivalent to the power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n+2} x^{n+1} \ ? \quad = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=3}^{\infty} a_{n+1} x^n$$

(a) $a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty}(a_n + a_{n+1})x^n$

(b) $a_0 + a_1 x + \sum_{n=2}^{\infty}(a_n + a_{n+3})x^n$

(c) $a_0 + a_1 x + \sum_{n=2}^{\infty}(a_n + a_{n+2})x^n$

(d) $a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty}(a_n + a_{n+2})x^n$

$= a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty}(a_n + a_{n+1})x^n$

10. Which one of the following is **a fundamental set of solutions** for the ODE

$$x^2y'' - 3xy' + 3y = 0, \quad x > 0 \ ?$$

(a) $f_1(x) = x$, $f_2(x) = x^2$

(b) $f_1(x) = x$, $f_2(x) = x^3$

(c) $f_1(x) = x^2$, $f_2(x) = x^3$

(d) $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = 2x + 3x^3$

$f_1' = 1, \ f_1'' = 0$

$\Rightarrow L.H.S = x^2(0) - 3x(1) + 3(x) = 0$ ✓

$f_2' = 3x^2, \ f_2'' = 6x$

$L.H.S = x^2(6x) - 3x(3x^2) + 3x^3$

$= 6x^3 - 9x^3 + 3x^3 = 0$ ✓

$W(y_1, y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3 \neq 0.$

11. The **Wronskian** of any two solutions of the ODE

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0 \text{ is}$$

(a) cx^2e^x

(b) $cx^{-2}e^{-x}$

(c) cxe^x

(d) cxe^{-x}

$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 0$

$W = ce^{-\int p(x)\,dx} = ce^{\int \frac{x+2}{x}dx} = ce^{\int(1 + \frac{2}{x})dx}$

$= ce^{x + 2\ln x}$

$= ce^x \cdot x^2$

12. Consider the following ODE

$$y'' + 2\alpha y' + y = 0.$$

Assume that its characteristic equation has complex roots. Then

$$\lim_{t\to\infty} y(t) = 0, \text{ if}$$

(a) $0 < \alpha < 1$

(b) $-1 < \alpha < 1$

(c) $-1 < \alpha < 0$

(d) $\alpha < -1$ or $\alpha > 1$

Aux. eq: $r^2 + 2\alpha r + 1 = 0$

$\Rightarrow r = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2}$

$= -\alpha \pm \sqrt{\alpha^2 - 1}$

$= -\alpha \pm \sqrt{1-\alpha^2}\, i$

$\Rightarrow y_h = c_1 e^{-\alpha t}\cos(\sqrt{1-\alpha^2}\, t) + c_2 e^{-\alpha t}\sin(\sqrt{1-\alpha^2}\, t)$

As the roots are complex $\Rightarrow 1 - \alpha^2 > 0$ or $\boxed{-1 < \alpha < 1}$

Also, since $\lim_{t\to\infty} y(t) = 0 \Rightarrow \boxed{+\alpha > 0}$

$\therefore \boxed{0 < \alpha < 1}$ ✓

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Question one (36%). Circle the correct answer.

1. Which one of the following is equivalent to the power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n+2} x^{n+1} \ ?$$

(a) $a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} (a_n + a_{n+2}) x^n$

(b) $a_0 + a_1 x + \sum_{n=2}^{\infty} (a_n + a_{n+3}) x^n$

(c) $a_0 + a_1 x + \sum_{n=2}^{\infty} (a_n + a_{n+2}) x^n$

(d) $a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} (a_n + a_{n+1}) x^n$

2. If $a < b < c < d$ are the roots of the auxiliary equation of the ODE

$$y^{(4)} + 3y^{(3)} - 4y'' - 12y' = 0,$$

then $a + 2b + c + d$ is equal to

(a) 2

(b) -1

(c) -5

(d) 5

3. Consider the following ODE

$$y'' + 9y' = t^2 \cos(3t).$$

How many terms are in the correct expression for the form of y_p?

(a) 6

(b) 4

(c) 2

(d) 8

4. The **Wronskian** of any two solutions of the ODE

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0 \text{ is}$$

(a) cxe^x

(b) $cx^{-2}e^{-x}$

(c) cx^2e^x

(d) cxe^{-x}

5. Consider the following ODE

$$y'' + 2\alpha y' + y = 0.$$

Assume that its characteristic equation has complex roots. Then

$$\lim_{t\to\infty} y(t) = 0, \text{ if}$$

(a) $\alpha < -1$ or $\alpha > 1$

(b) $-1 < \alpha < 1$

(c) $-1 < \alpha < 0$

(d) $0 < \alpha < 1$

6. Which one of the following non-homogeneous equations **can be solved** using the method of undetermined coefficients?

   (a) $ty'' - y = te^t$
   (b) $y^{(3)} + 8y' = \cos^4(t) - \sin^4(t)$
   (c) $3y^{(3)} - 2y'' + y' = \cos(t^2)$
   (d) $2y'' + 7y' - 13y = t^{-2}e^{-2t}, \ t > 0.$

7. Suppose that y_1 and y_2 are solutions of the **nonhomogeneous** differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

   and y_3 and y_4 are solutions of the corresponding **homogeneous** differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

   One of the following statements is **false.**

   (a) $5y_3$ is a solution of the homogeneous differential equation
   (b) $y_1 - y_3$ is a solution of the nonhomogeneous differential equation
   (c) $y_3 + y_4$ is a solution of the homogeneous differential equation
   (d) $y_3 - y_1$ is a solution of the nonhomogeneous differential equation

8. The **radius of convergence** of the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n \text{ is equal to}$$

   (a) 0
   (b) ∞
   (c) 2
   (d) 1

9. Let $y(t)$ be the solution of the IVP:

$$y'' - 2y' + y = 0, \;\; y(0) = y'(0) = 1.$$

Then $y(\ln 2) =$

(a) $\frac{1}{2}$
(b) $2\ln 2$
(c) 2
(d) $\frac{3}{4}$

10. Which one of the following is **a fundamental set of solutions** for the ODE

$$x^2y'' - 3xy' + 3y = 0 \;\; ?$$

(a) $f_1(x) = x^2,\; f_2(x) = x^3$
(b) $f_1(x) = x,\; f_2(x) = x^3$
(c) $f_1(x) = x,\; f_2(x) = x^2$
(d) $f_1(x) = x,\; f_2(x) = x^3,\; f_3(x) = 2x + 3x^3$

11. **The form of** y_p of the following nonhomogeneous linear differential equation

$$y^{(3)} - 12y' - 16y = e^{-2t} - te^{4t} \;\; \text{is}$$

(a) $y_p = At^2e^{-2t} + Bt^2e^{4t} + Cte^{4t}$
(b) $y_p = Ae^{-2t} + Bte^{4t} + Ce^{4t}$
(c) $y_p = Ate^{-2t} + Bte^{4t}$
(d) $y_p = At^2e^{-2t} + Bte^{-2t} + Ct^2e^{4t}$

12. The solution of the boundary-value problem

$$2x^2y'' + 3xy' - y = 0,\; y(1) = 2,\;\; y(4) = \frac{9}{4}, \;\; \text{satisfies } y(9) =$$

(a) $\frac{1}{9}$
(b) $\frac{28}{9}$
(c) $\frac{1}{3}$
(d) $\frac{7}{9}$

Question Two (14 %). (a) Solve the following IVP

$$y'y'' = t, \; y(1) = 2, \; y'(1) = 1.$$

(8 points)

let $v = y' \Rightarrow v v' = t$

or $\int v \, dv = \int t \, dt$

(4 pts)

$\Rightarrow \boxed{\frac{v^2}{2} + c = \frac{t^2}{2}}$

$\frac{[v(1)]^2}{2} + c = \frac{1}{2} \Rightarrow \frac{1}{2} + c = \frac{1}{2} \Rightarrow \boxed{c = 0}$

$\therefore \frac{v^2}{2} = \frac{t^2}{2} \Rightarrow \boxed{v = t}$ (we take +ve since $y'(1) = 1 > 0$)

(4 pts)

$\Rightarrow \int dy = \int t \, dt \Rightarrow \boxed{y = \frac{t^2}{2} + k}$

$y(1) = 2 \Rightarrow 2 = \frac{1}{2} + k \Rightarrow k = 3/2$

$\therefore \boxed{y = \frac{t^2}{2} + \frac{3}{2}}$.

(b) Without the use of the Wronskian, show that the functions

$$f_1(x) = \cos^2(x)(1 + \tan x)^2, \; f_2(x) = 2018, \; \text{and} \; f_3(x) = \sin(2x)$$

are linearly dependent on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(6 points)

3 pts

$f_1 = [\cos x (1 + \tan x)]^2$

$= (\cos x + \sin x)^2$

$= \cos^2 x + \sin^2 x + 2 \sin x \cos x$

$= 1 + \sin(2x)$

3 pts

$= \frac{1}{2018} \cdot 2018 + \sin 2x$

$\Rightarrow \boxed{f_1 = \frac{1}{2018} f_2 + f_3}$

$\therefore \{f_1, f_2, f_3\}$ are lin. dep.

Question Three (20 %). **(a)** The function $y_1 = e^{-2x}$ is a solution of the ODE

(10 pts)

$$(1+2x)y'' + 4xy' - 4y = 0.$$

Use the **Reduction of Order Formula,** to find a second linearly independent solution y_2.

$$y'' + \underbrace{\frac{4x}{1+2x}}_{p(x)} y' - \frac{4}{1+2x} y = 0$$

Long division

(4 pts)

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx = e^{-2x} \int \frac{e^{-\int \left(\frac{4x}{2x+1}\right)dx}}{e^{-4x}}\,dx$$

(3 pts)

$$= e^{-2x} \int \frac{e^{-\int \left(2 - \frac{2}{2x+1}\right)dx}}{e^{-4x}}\,dx$$

$$= e^{-2x} \int e^{-2x + \ln|2x+1|} \cdot e^{4x}\,dx$$

$$= e^{-2x} \int e^{-2x} \cdot (2x+1)\, e^{4x}\,dx, \quad x > -\frac{1}{2}$$

$$= e^{-2x} \int (2x+1)\, e^{2x}\,dx$$

$2x+1$	e^{2x}
2 $\xrightarrow{(+)}$	$\frac{1}{2}e^{2x}$
0 $\xrightarrow{(-)}$	$\frac{1}{4}e^{2x}$

(3 pts)

$$= e^{-2x}\left[\left(x+\tfrac{1}{2}\right)e^{2x} - \tfrac{1}{2}e^{2x}\right]$$

$$= x + \frac{1}{2} - \frac{1}{2} = x$$

$$\Longrightarrow \boxed{y_2 = x}$$

$$\therefore\; y_H = c_1 e^{-2x} + c_2 \cdot x$$

(10 pts) **(b)** Solve the following nonhomogeneous differential equation

$$y^{(4)} + y'' + y = e^{2t}.$$

For y_h: aux. eq $r^4 + r^2 + 1 = 0$

(3 pts)

$$\Rightarrow \underbrace{r^4 + 2r^2 + 1} - r^2 = 0$$

$$\Rightarrow (r^2+1)^2 - r^2 = 0$$

$$\Rightarrow (r^2+1-r)(r^2+1+r) = 0.$$

$$\Rightarrow r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

(2 pts)

$$\Rightarrow \boxed{y_h = c_1 e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_4 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).}$$

For y_p: let $\boxed{y_p = Ae^{2t}}$ (2 pts) $\Rightarrow y_p' = 2Ae^{2t}$

$y_p'' = 4Ae^{2t}$, $y_p''' = 8Ae^{2t}$

$y_p^{(4)} = 16Ae^{2t}$

Substitute y_p, y_p', y_p'', y_p''' & $y_p^{(4)}$ into the d.e,

$$16Ae^{2t} + 4Ae^{2t} + Ae^{2t} = e^{2t}$$

$$\Rightarrow 21A = 1 \Rightarrow \boxed{A = \frac{1}{21}}$$

(2 pts) $\therefore \boxed{y_p = \frac{1}{21}e^{2t}}$

Hence, $y_g = y_h + y_p$, where y_c & y_p as above.

(1 pts)

Question Four (10 %). **(a)** Let $y(x) = e^{3x}(c_1 + c_2\cos 2x + c_3\sin 2x)$ be general solution of a homogeneous third order linear differential equation with constant coefficients. Find the corresponding differential equation.

$y = c_1 e^{3x} + c_2 e^{3x}\cos 2x + c_3 e^{3x}\sin 2x$

(2pts) by inspection, the roots of its auxilliary eq. are

$r = 3, \quad 3 \pm 2i$

Now, $r = 3 \Rightarrow (r-3)$ is a factor

$r = 3 \pm 2i \Rightarrow (r-3)^2 = -4 \Rightarrow (r^2 - 6r + 13)$ is a factor

(2pts) $\therefore$ aux. eq. $(r-3)(r^2-6r+13) = 0$

$\Rightarrow r^3 - 6r^2 + 13r - 3r^2 + 18r - 39 = 0.$

$\Rightarrow r^3 - 9r^2 + 31r - 39 = 0$

$\therefore$ d.e $\boxed{y''' - 9y'' + 31y' - 39y = 0}$ (1 pts)

(b) What is the **Cauchy-Euler homogeneous** linear differential equation of second order whose general solution is

$$y = \frac{1}{\sqrt{x}}(k_1 + k_2 \ln x), \quad x > 0 \ ?$$

$y = k_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$

2 pts $\Rightarrow$ roots of the aux. eq. are $m = -\frac{1}{2}, -\frac{1}{2}$

$\Rightarrow (2m+1)^2 = 0$ is the eq.

(1 pts) or $4m^2 + 4m + 1 = 0$

a   b−a   c

$\Rightarrow \boxed{a = 4}, \quad b - a = 4 \Rightarrow \boxed{b = 8}, \quad \boxed{c = 1}$

$\therefore$ the d.e is $\boxed{4x^2y'' + 8xy' + y = 0}$

(2pts)

Question Five (10%). Find the general solution of the differential equation

$$y^{(5)} + 3y^{(4)} - 5y^{(3)} + 17y'' - 36y' + 20y = 0,$$

if it is known that $y_1 = te^t$ is one solution.

(1 pts) Since $y_1 = te^t$ is one solution $\Rightarrow r_1 = r_2 = 1$ are roots of the aux. eq.

(3 pts)

1	1	3	-5	17	-36	20
		1	4	-1	16	-20
1	1	4	-1	16	-20	0
		1	5	+4	20	
	1	5	+4	20	0	

$\therefore$ the aux. eq. is

$$(r-1)^2(r^3+5r^2+4r+20) = 0.$$

$$\Rightarrow (r-1)^2\left(r^2(r+5)+4(r+5)\right)$$ (3 pts)

$$\Rightarrow (r-1)^2(r+5)(r^2+4) = 0.$$

$$\Rightarrow r = 1, 1, -5, \pm 2i$$ (1 pts)

$$\Rightarrow \boxed{y_h = c_1e^t + c_2te^t + c_3e^{-5t} + c_4\cos(2t) + c_5\sin(2t)}.$$

(2 pts)

Question Six (10%). Use the method of **variation of parameters** to solve

$$y'' - 2y' + y = \frac{e^t}{1+t^2}.$$

(2 pts) step1 y_h aux. eq. $r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$

$\therefore \boxed{y_h = c_1 e^t + c_2 t e^t} \Rightarrow \boxed{y_1 = e^t}, \boxed{y_2 = t e^t}$

(2 pts) step2 $W(y_1, y_2) = \begin{vmatrix} e^t & t e^t \\ e^t & t e^t + e^t \end{vmatrix} = \cancel{t e^{2t}} + e^{2t} - \cancel{t e^{2t}}$

$= e^{2t}.$

step3 $y_p = v_1 y_1 + v_2 y_2$, where

(1.5 pts) $v_1 = -\int \frac{g \cdot y_2}{W} dt = -\int \frac{\frac{e^t}{1+t^2} \cdot t e^t}{e^{2t}} dt.$

$= -\int \frac{t}{1+t^2} dt = -\frac{1}{2} \ln(1+t^2).$

(1.5 pts) $v_2 = \int \frac{g \cdot y_1}{W} dt = \int \frac{\frac{e^t}{1+t^2} \cdot e^t}{e^{2t}} = \int \frac{1}{1+t^2} dt$

$= \tan^{-1} t.$

(2 pts) $\therefore y_p = v_1 y_1 + v_2 y_2$

$\boxed{y_p = -\frac{1}{2} e^t \ln(1+t^2) + t e^t \tan^{-1} t}.$

Good Luck

(1 pts) $\therefore y_g = y_h + y_p$

$= c_1 e^t + c_2 t e^t - \frac{1}{2} e^t \ln(1+t^2) + t e^t \tan^{-1} t.$

وفقكم الله