2.6 Exact Equations And integrating factors

If you have an Equation on the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Then : • The Eq is Exact if $M_y = N_x$

• يعني اشتق معامل dx بالنسبة لـ y ومعامل dy بالنسبة لـ x وافحص المساواة
• اذا كانت النتيجة ايجابية اذن نستطيع ايجاد الحل

$$\Psi(x,y) = c = \int \Psi_x\,dx$$

• الحل عكن ايجاده بتكامله Ψ_x بالنسبة لـ x ثم مساواته بـ Ψ_y وايجاد $g(y)$ بعد اشتقاقها (التوضيح في المثال)

Ex:- Outline 7,

$$\underbrace{(e^x\sin y - 2y\sin x)}_{M}\,dx + \underbrace{(e^x\cos y + 2\cos x)}_{N}\,dy = 0$$

• Step 1: Derive and Compare

مشتقة M بالنسبة لـ y

$$M = e^x\sin y - 2y\sin x \longrightarrow M_y = e^x\cos y - 2\sin x$$

$$N = e^x\cos y + 2\cos x \longrightarrow N_x = e^x\cos y \mp 2\sin x$$

مشتقة N بالنسبة لـ x

$$M_y = N_x \qquad \text{(it's Exact)}$$

• Step 2: find Ψ_x and Ψ_y and $\Psi_{(x,y)}$

$$\Psi_x = M = e^x\sin y - 2y\sin x$$

$$\Psi_y = N = e^x\cos y + 2\cos x$$

Now:-

$$\psi_{(x,y)} = \int \psi_x \, dx = \int e^x \sin y - 2y\sin x \, dx$$

$$= e^x \sin y + 2y\cos x + g(y)$$

to find $g(y)$:-

Step 3: $(\psi_y) = (\psi_{(x,y)})'$ مشتقة بالنسبة لـ y

$$e^x\cos y + 2\cos x = e^x\cos y + 2\cos x + g'(y)$$

so $g'(y) = 0$

$g(y) = 0$

• The Solution is :- $\psi_{(x,y)} = c$

$$e^x \sin y + 2y\cos x = c$$

• Non-Exact Equations transformation into Exact Equations

Integrating factors :-

* if $M_y \neq N_x$ Then :-

We find: $\frac{M_y - N_x}{N}$

if It's a function of x Then: $= f(x)$

• $I(x) = e^{\int f(x)dx}$

if it's a function of y Then $= g(y)$

• $I(y) = e^{\int f(y)dy}$

• Multiply The Eq with $I(x)$ or $I(y)$ To Make it Exact

Not (Outline) : 25

$$\underbrace{(3x^2y+2xy+y^3)}_{M}dx+\underbrace{(x^2+y^2)}_{N}dy=0$$

$M=3x^2y+2xy+y^3 \longrightarrow M_y=3x^2+2x+3y^2$

$N=x^2+y^2 \longrightarrow N_x=2x$

It's Not Exact :-

Check:- $\dfrac{M_y-N_x}{N}=\dfrac{3x^2+2x+3y^2-2x}{x^2+y^2}=\dfrac{3x^2+3y^2}{x^2+y^2}=3$

find I:-

$I(x)=e^{\int 3\,dx}=e^{3x}$

The Sol is:- $e^{3x}x^2y+\dfrac{e^{3x}y^3}{3}=c$

NoW:-

$e^{3x}(3x^2y+2xy+y^3)dx+e^{3x}(x^2+y^2)dy=0$

$\psi_x=e^{3x}(3x^2y+2xy+y^3)$

Not By Parts Because $\int$ With Respect to y

$\psi_y=e^{3x}(x^2+y^2)$

$\psi_{(x,y)}=\int \psi_y\,dy=\int(e^{3x}(x^2)+e^{3x}y^2)dy$

$=e^{3x}x^2y+\dfrac{e^{3x}y^3}{3}+g(x)$

Derive With Respect to X :- And $\stackrel{?}{=}\psi_x$

$\cancel{e^{3x}2xy}+x^2y(3)(e^{3x})+\cancel{\dfrac{3e^{3x}y^3}{3}}+g'(x)=e^{3x}3x^2y+\cancel{e^{3x}2xy}+\cancel{e^{3x}y^3}$

so $g'(x)=0 \rightarrow g(x)=0$