2.8: The existance and Uniqueness Theorem

• Picard's Method :- The Method of successive approximation
~~to Solve~~ with this Method you need to find $\phi_n(t)$ and check weather it converges or not

If it Converges → There is a Solution for The IVP : $y = \phi(t)$

If it Diverges → There is No info about your prob.

How to find $\phi(t)$?

$$y = \phi(t) = \int_0^t f(s, \phi s)\, ds$$

Usually ϕ_n is a Maclaurin Series (Not always)

to find $\phi_n(t)$:-
find

$$\phi_1(t) = \int_0^t f(s, \phi_0 s)\, ds$$

$$\phi_2(t) = \int_0^t f(s, \phi_1 s)\, ds$$

$$\vdots$$

$$\phi_n(t) = \int_0^t f(s, \phi_{n-1} s)\, ds$$

* الصيغة العامة
بأخذ لها ال limit.

• Test for Convergence Using Series Test that we Took in CalCulus 2

How to Solve Questions ?

Outline 3:-

$$\begin{cases} \frac{dy}{dt} = 2(y+1) \\ y(0) = 0 \end{cases}$$ ← always Given

Step 1: find ϕ_n :-

$f(t,y) = 2(y+1)$

$\phi_0(t) = 0$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$$

$$= \int_0^t f(s, 0)\, ds$$

$$= \int_0^t 2(1)\, ds = 2t$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$= \int_0^t f(s, 2s)\, ds$$

$$= \int_0^t 2(2s+1)\, ds$$

$$= \int_0^t (4s + + 2)\, ds = \frac{4s^2}{2} + 2s \Big]_0^t$$

$$= 2t^2 + 2t$$

$$\phi_3(t) = \int_0^t f(s, \phi_2(s))\, ds$$

$$= \int_0^t f(s, 2s^2+2s)\, ds$$

$$= \int_0^t 2(2s^2+2s+1)\, ds = \frac{4s^3}{3} + \frac{4s^2}{2} + 2s \Big]_0^t$$

* transforming initial Value problems that start ⓐ at a point to an equivalent problem with the initial Point at the Origin

Ex.- Outline (1):-

$$\text{IVP}\begin{cases} \dfrac{dy}{dt} = \cancel{2}t^2 + y^2 \\ y(1) = 2 \end{cases}$$

$\rightsquigarrow$ You need to find a new function to make this point $\underline{w(0) = a}$

Now :

let's say The new function is $w(s) = y(t) - a$

$$y(t) - a = 0$$
$$y(t) = a$$
$$y(1) = 2$$
$$\text{so } a = 2 \rightsquigarrow w = y(t) - 2$$

So $(w+2) = y(t)$ ①

$$w(s) = y(t) - 2$$

now s should be a functiof of t and equals zero at t=1

So $s = t - 1 \Rightarrow \underline{t = s+1}$ ②

$$\frac{d(s-1)}{dt} = 1$$

Now : $$\frac{dy}{dt} = \frac{dw}{ds} \cdot \frac{ds}{dt} = \frac{dw}{ds}$$

So $$\frac{dy}{dt} = \frac{dw}{ds}$$

The New IVP
$$\begin{cases} \boxed{\dfrac{dw}{ds} = (s+1)^2 + (w+2)^2} \\ w(0) = 0 \end{cases}$$

$= \frac{4t^3}{3} + 2t^2 + 2t \quad = \frac{8}{6}t^3 + \frac{4}{2}t^2 + 2t$

$\phi_n = \sum_{i=1}^{\infty} \frac{2^i t^i}{i!} \rightarrow$ You have to check if it Converges or not

Step 2: Use Ratio test :-

$$\lim_{n\to\infty} \frac{2^{n+1} t^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n t^n}$$

$$= \lim_{n\to\infty} \frac{2t}{n+1} = 0<1$$

So It Converges :-

The Solution for the IVP is :-

$$y = \phi(t) = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} = \frac{(2t)^0}{0!} + \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} = e^{2t}$$

$$y = 1 - e^{2t}$$