Chap 3: Second order linear Equations

3.1 + 3.3 Homogenous Equations with Constant Coefficients

General form: $y'' = f(t, y, y')$

$$y'' + p(t)y' + q(t)\,y = g(t)$$

$g(t) = 0$ Homogenous

$g(t) \neq 0$ non Homogenous

→ Now for $p(t)$, $q(t)$ are Constants and $g(t) = 0$

- Steps for Sol:-

1- $ay'' + by' + cy = 0$

2- assume a Characteristic Equation:-

$ar^2 + br + c = 0 \Rightarrow$ The Sol is : $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3- There are three Cases:-

$\boxed{1}$ $r_1 \neq r_2$, real $\Rightarrow y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$\boxed{2}$ $r_1 = r_2$, real $\Rightarrow y_h = C_1 e^{rt} + C_2 t e^{rt}$

$\boxed{3}$ $r = \alpha \pm i\beta$, Complex $\Rightarrow y_h = C_1 e^{\alpha t} \cos\beta t + C_2 e^{\alpha t} \sin\beta t$ 3.3

↳ Remember: $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

Example:-

(15) outline :-

aux Eq: $y'' + 8y' - 9y = 0$ $\qquad$ $y(1) = 1$, $y'(1) = 0$ — Q

$$r^2 + 8r - 9 = 0$$

$$r_{1,2} = \frac{-8 \pm \sqrt{64 - (4)(-9)}}{2}$$

find Solutions (Roots)

$$r_{1,2} = \frac{-8 \pm 10}{2} \Rightarrow \left.\begin{array}{l} r_1 = 1 \\ r_2 = -9 \end{array}\right\} \text{real and Distinct}$$

$$y_h = c_1 e^{t} + c_2 e^{-9t}$$

$y(1) = 1 \rightarrow$ $\quad 1 = c_1 e + c_2 e^{-9}$ — ①

• find Derivative of y_h :-

$$y_h' = c_1 e^{t} - 9 c_2 e^{-9t}$$

$y_h'(1) = 0$

$$0 = c_1 e - 9 c_2 e^{-9}$$ — ②

put ② and ①

$$1 = 9 c_2 e^{-9} + c_2 e^{-9}$$

$$\frac{1}{10 e^{-9}} = c_2 \rightsquigarrow c_2 = \frac{e^9}{10}$$

find c_1 :- $\quad c_1 e = \frac{9}{10} e^9 e^{-9}$

$$c_1 = \frac{9e^{-1}}{10}$$

Sol :-

$$y_h = \frac{9e^{t-1}}{10} + \frac{e^{9-9t}}{10}$$

3.2 Solutions of linear Homogenous Equations (The Wronskian)

→ There are 6 Theories

Th. 1:- Existance and Uniqueness Them

$$y'' + p(t)y' + q(t)y = g(t)$$

functions Cont on an open Interval $I(\alpha, \beta)$ Containing t_0

↳ Then the IVP has Exactly one Solution.

Outline 8:-

$$(t-1)y'' - 3ty' + 4y = \sin t \qquad t_0 = -2$$

$$y'' - \frac{3t}{t-1}y' + \frac{4}{t-1}y = \frac{\sin t}{(t-1)}$$

$p(t)$ ↓ cont on $(-\infty, 1) \cup (1, \infty)$

$q(t)$ ↓ cont on $(-\infty, 1) \cup (1, \infty)$

$g(t)$ ↓ cont on $(-\infty, 1) \cup (1, \infty)$

so The largest interval is $(-\infty, 1)$

Th.2: Principle of Superposition

$L[y]$: differential Operator

→ y_1, y_2 Sol of The d.e. $y'' + p(t)y' + q(t)y = 0$

Then $C_1y_1 + C_2y_2$ is also a Sol

Th.3: FTh 4

find Woronskian for y_1, y_2

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Imp

If $W(y_1, y_2) \neq 0$ ⇝ $y = c_1y_1 + c_2y_2$ Satisfies $L[y]=0$

and y_1, y_2 are a fundemental set of solutions of Eq $L[y] = y'' + P(t)y' + q(t)y = 0$ —— *

- y_1 and y_2 are linearly independant on I (open Interval) iff $W(y_1, y_2)(t) \neq 0$ for at least $t \in I$

* To Know wheather y_1 and y_2 are fundumental solutions:-

→ make sure that $W(y_1, y_2) \neq 0$ and That y_1, y_2 satisfies Eq (*)

Example : Outline 24

$y'' + 4y = 0$ —— ①

$y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$

Step 1:- make sure that each function satisfies the Eq ①

$y_1(t)$:-

$y_1(t) = \cos 2t$

$y_1'(t) = -2\sin 2t$

$y_1''(t) = -4\cos 2t$ Put in ①

L.H.S $-4\cos 2t + 4\cos 2t = 0 = $ R.H.S

$y_2(t)$:-

$y_2(t) = \sin 2t$

$y_2'(t) = 2\cos 2t$

$y_2''(t) = -4\sin 2t$

L.H.S $-4\sin 2t + 4\sin 2t = 0 = $ R.H.S

So Condition 1 is satisfied ☑

Step 2:

$$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t$$

$$= 2(\cos^2 2t + \sin^2 2t)$$

1

$= 2 \neq 0$

So Condition 2 is satisfied ☑ Then The set is a fundamental Sol

Them : 3.2.6 Abel's form

$y_1, y_2 \longrightarrow$ Sol's of $L[y] = y'' + p(t)y' + q(t) = 0$

Then $W(y_1, y_2) = Ce^{-\int p(t)\,dt}$

Depends on y_1, y_2

$c = 0$ — $W = 0 \ \forall t \in I$

$c \neq 0$ — $W \neq 0 \ \forall t \in I$

Ex Outline 29

find W without Solving the equation:

$$t^2 y'' - t(t+2)\, y' + (t+2)\, y = 0$$

Using Abel's Them:-

$$y'' - \frac{t(t+2)}{t^2}\, y' + \frac{(t+2)}{t^2}\, y = 0$$

$$W = ce^{-\int -\frac{t+2}{t}\,dt}$$

$$= ce^{\int 1 + \frac{2}{t}\,dt} = Ce^{(t + 2\ln t)}$$

$$= Ce^{t + \ln t^2}$$

$$\boxed{W = ce^{t}\, t^2}$$

3.3 Continue

If Roots of the aux Eq. is Complex numbers

Then :-

The General Sol :- $y_h = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$

→ The Roots :- $r_{1,2} = \alpha \pm \beta i$

Euler formula

If you have an equation : $t^2 \frac{d^2y}{dt^2} - \alpha t \frac{dy}{dt} - \beta y = 0$

You Can re write it at the form :-

$$y'' + (\alpha - 1) y' + \beta y = 0$$

$$\frac{d^2y}{dx^2} + (\alpha - 1) \frac{dy}{dx} + \beta y = 0 \quad \text{where } x = \ln t$$

→ This Easier to Solve

Example Outline 38

$$t^2 y'' - 4t y' - 6y = 0$$

$$\alpha = -4 \quad , \quad \beta = -6$$

So The E. formulas is :- $y'' - 5y - 6 = 0$

The aux : $r^2 - 5r - 6 = 0$

$$r_{1,2} = \frac{-5 \pm \sqrt{25+24}}{2} = \frac{5 \pm 7}{2}$$

$$r_1 = 6 \ , \ r_2 = -1$$

Sol is:

$$y_h = c_1 e^{6x} + c_2 e^{-x}$$

$$y_h = c_1 e^{6 \ln t} + c_2 e^{-\ln t}$$