Math 331. Chapter 1. Introduction

1.3 Classification of Differential equations.

<u>Def:</u> Differential equations are relation containing derivatives.

DEs

- ODEs (ordinary differential equations)
- PDEs (partial differential equation)
- system of DEs

1) Ordinary differential Equations (ODEs).

The unknown function depends on one independent variable and only ordinary derivatives appear in the equation.

<u>Example 1.</u> $\dfrac{dv}{dt} = 9.8 - \frac{1}{5}v$ $\quad (v = v(t))$

t: Independent
v: dependent

Example 2. $\dfrac{dP}{dt} = \dfrac{1}{2}P - 450$ is ODE. $(P = P(t))$.

Example 3. $\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} + y = x^2$. $(y = y(x))$

2) Partial Differential equations (PDEs).

The unknown function depends on two or more independent variables and partial derivatives appear in the equation

Example (a). $\alpha^2 U_{xx} = U_t$ or $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$.

This equation is called heat equation.

Example (b). $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ (Wave equation).

3) System of Differential Equations

Two or more unknown functions require a system of Differential equations.

Example. (Lotka-Voltera) equation.

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy \\[2ex] \dfrac{dy}{dt} = -cy + \gamma xy \end{cases} \quad , \quad (x = x(t) \;, y = y(t)) \quad , \quad a, \alpha, c, \gamma \in \mathbb{R}$$

Def: The order of a D.E is the order of the highest derivative of the unknown function that appears in the equation.

Example: 1. $\dfrac{dy}{dt} - ty = t^3$. (1st order ODEs).

Example 2. $\left(\dfrac{d^2 q}{dx^2}\right)^5 + \cos(x+q) = 0$. (2nd order ODE).

Linear and nonlinear DEs.

The ODE $F(t, y, y', \dots, y^{(n)}) = 0 \; \cdots (*)$ is said to be linear if F is a linear function of the variables $y, y', \dots, y^{(n)}$.

Thus, the general linear ordinary D.E of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_n(t)\, y = g(t) \quad \dots (**)$$

An equation that is not of the form $(**)$ is a nonlinear equation.

Example: Classify the following DEs.

(1) $y' - 2y = t^3$. 1st order linear ODE.

(2) $t^2 y'' + t y' + (\sin t) y = 0$. 2nd order linear ODE.

(3) $\frac{dp}{dt} + t p^2 = \cos t$. 1st order Nonlinear ODE.

(4) $\dfrac{d^2 q}{dx^2} + \cos(x+q) = 0.$ 2nd order Nonlinear ODE.

(5) $\dfrac{d^3 x}{dy^3} + \left(\dfrac{d^2 x}{dy^2}\right)^5 + y^6 = x.$ 3rd order Nonlinear ODE.

(6) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^3 u}{\partial x^2 \partial y} = x^2 + y^2.$ 3rd order linear PDE.

(7) $(x+e^y)\,dy = dx \iff \dfrac{dy}{dx} = \dfrac{1}{x+e^y}.$

1st order Nonlinear in y ODE.

But : $\dfrac{dx}{dy} = x + e^y.$ 1st order linear in x ODE

Def: A solution of the ODE (*), on the Interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy

$$F(t, \phi, \phi', \dots, \phi^{(n)}) = 0 \quad , \forall t \in (\alpha, \beta).$$

Example: Verify that $y = 3x + x^2$ is a Solution

of the D.E $x\dfrac{dy}{dx} - y = x^2$.

Sol: $\dfrac{dy}{dx} = 3 + 2x$.

L.H.S : $x\underbrace{(3+2x)}_{\frac{dy}{dx}} - \underbrace{(3x+x^2)}_{y} = 3x + 2x^2 - 3x - x^2 = x^2$

R.H.S.

Example: Verify that $y = (\cos t)\ln(\cos t) + t\sin t$

is a Solution of the ODE.

$$y'' + y = \sec t \quad , 0 < t < \frac{\pi}{2}$$

Sol: $y' = \cos t \cdot \left(\dfrac{-\sin t}{\cos t}\right) - \sin t \ln(\cos t) + t\cos t + \sin t.$

$$y'' = -\cancel{\cos t} - \left[\sin t\left(\frac{-\sin t}{\cos t}\right) + \cos t \ln(\cos t)\right]$$

$$+ \left[-t\sin t + \cos t\right] + \cancel{\cos t}.$$

$$\Rightarrow y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t\sin t.$$

L.H.S : $y'' + y =$

$$= \cancel{-\cos t \ln(\cos t)} + \frac{\sin^2 t}{\cos t} + \cos t \cancel{- t\sin t} + \left(\cancel{\cos t \ln(\cos t)} + \cancel{t \sin t}\right)$$

$$= \frac{\sin^2 t}{\cos t} + \cos t = \frac{\sin^2 t + \cos^2 t}{\cos t}$$

$$= \frac{1}{\cos t} = \sec t = \text{R.H.S.}$$

Home Work # 1 :

1) Verify that $y = e^{t^2}\int_0^t e^{-r^2}\,dr + e^{t^2}$ is a solution of $y' - 2ty = 1$.

2) Verify that $y = \ln x / x^2$ is a solution of $x^2 y'' + 5x y' + 4y = 0$, $x > 0$.

1.1 Some basic Models and Direction Fields.

Example 1. Suppose that an object is falling in the atmospher near sea level. Formulate a differential equation that describes the motion.

Sol: Let t : time (Independent variable).

v : Velocity of the falling object. (dependent variable).

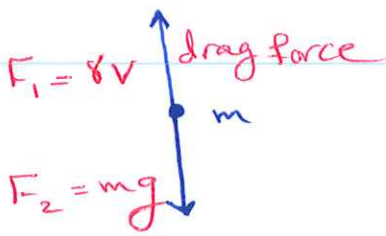
Using Newton's second law : $\boxed{F = ma}$, where

m : mass of the object.

a : acceleration.

F : The net force exerted on the object.

$F_{net} = F_2 - F_1$



$\Rightarrow ma = mg - \gamma v$

$\Rightarrow m \dfrac{dv}{dt} = mg - \gamma v$

or $\boxed{\dfrac{dv}{dt} = g - \dfrac{\gamma}{m} v}$ ... ①

(1st order linear ODE).

Where g: The acceleration due to gravity.

γ : drag Coefficient.

v : velocity.

Remark : To solve Eq. (1), we need to find a function $v = v(t)$ that satisfies the equation. (section 1.2)

To investigate the Behaviour of the Solution of Eq. (1) without solving it, we will use "direction field" or "Slope field".

To draw the direction field for eq. (1), take $m = 10$ kg, $\gamma = 2$ kg/s. In this case:

eq (1) becomes:

$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}} \quad \cdots (2)$$

Now, we find the equilibrium Solution of D.E (2) by setting $\frac{dv}{dt} = 0$, this implies

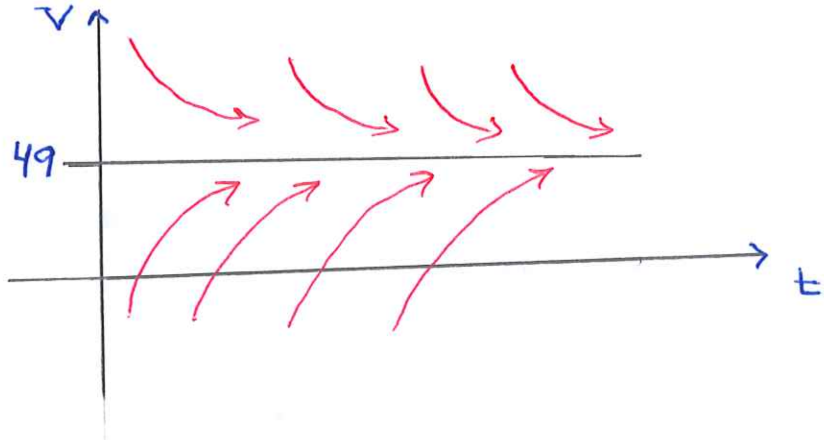
$$9.8 - \frac{V}{5} = 0 \implies \boxed{V = 49} \text{ "Equilibrium Sol."}$$

Next, we choose values for V below 49.

Take $V = 5 \Rightarrow \frac{dv}{dt} = 9.8 - 1 = 8.8 > 0$.

Then choose values for V above 49.

Tak $V = 80 \Rightarrow \frac{dv}{dt} = 9.8 - \frac{80}{5} = -6.2 < 0$



Remark: Solutions below the equilibrium solution ($V=49$) increase with time, and those above it decrease with time and all other solutions approach ($V=49$).

That is $\lim_{t\to\infty} V(t) = 49$

Example: Draw a direction field for the given D.E, then determine the behaviour of y as $t \to \infty$.

① $\dfrac{dy}{dt} = 2y + 3.$

• Equilibrium: $2y + 3 = 0 \implies \boxed{y = -\frac{3}{2}}$

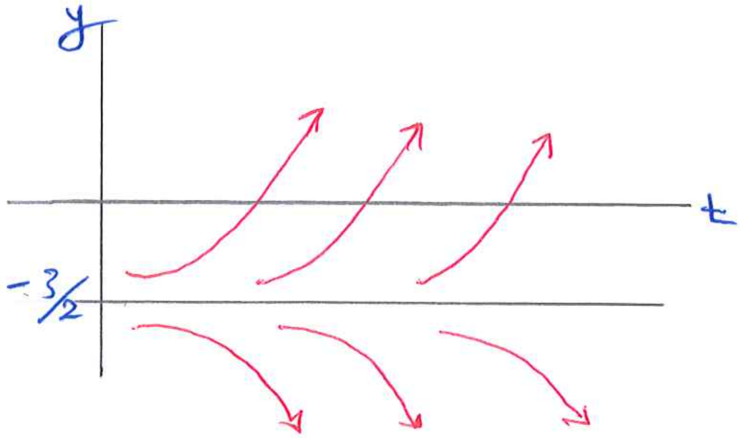
If $y_0 > -\frac{3}{2}$, (Take $y_0 = 0$), then

$\dfrac{dy}{dt} = 3 > 0$

If $y_0 < -\frac{3}{2}$, (take $y_0 = -2$), then $\dfrac{dy}{dt} = -1 < 0.$

$\Rightarrow$ Behaviour: $\lim\limits_{t \to \infty} y(t) = \begin{cases} +\infty & , y_0 > -3/2 \\ -\infty & , y_0 < -\frac{3}{2} \end{cases}$

Therefore, the solution diverges from $-\frac{3}{2}$ as $t \to \infty$



② $y' = y(y-1)^2$.

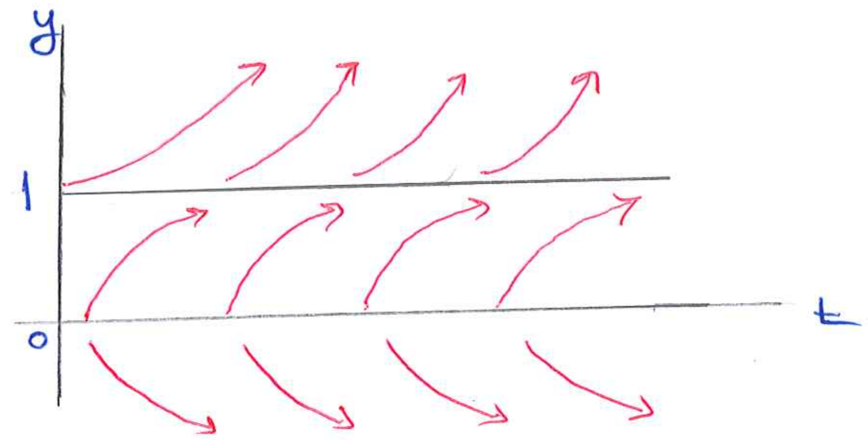
Equilibrium

$\frac{dy}{dt} = 0 \Rightarrow y(y-1)^2 = 0 \Rightarrow \boxed{y=0}$ or $\boxed{y=1}$

If $y_0 < 0$, take $y_0 = -1$, then $y' < 0$

If $0 < y_0 < 1$, take $y_0 = 0.5$, then $y' > 0$

If $y_0 > 1$, take $y_0 = 2$, then $y' > 0$



That means: If the initial value is negative, ($y_0 < 0$), then y diverges from 0 as $t \to \infty$

If the Initial value is between 0 and 1, then $y \to 1$ as $t \to \infty$

If the initial value is greater than 1, then y diverges from 1 as $t \to \infty$

③ $y' = y(y-1)^2$, $y(0) = 2023$

From example ② , y diverges. , $\lim_{t \to \infty} y(t) = +\infty$

④ $\begin{cases} y' = y(y-1)^2 \\ y(0) = 0.1 \end{cases}$

From example ② , $\lim_{t \to \infty} y(t) = 1$.

Remark : A differential equation together with initial Condition is called initial value Problem (IVP).

like example ③ & ④.

Example: Field Mice and Owls.

Consider a population of field mice who inhabit a certain rural area. Assume that the mouse population increases at a rate proportional to the current population. The D.E that describes the growth $p(t)$ is

$$\frac{dp}{dt} = rp \quad \cdots (3)$$

where r : rate constant or growth rate.

$p(t) = p$: population of mice field at time t

t : time.

Example: Assume that $r = 0.5/\text{month}$ and owls are present and they kill 15 field mice per day.

So the D.E (3), becomes:

$$\frac{dp}{dt} = \frac{1}{2}p - 450. \quad \cdots (4)$$

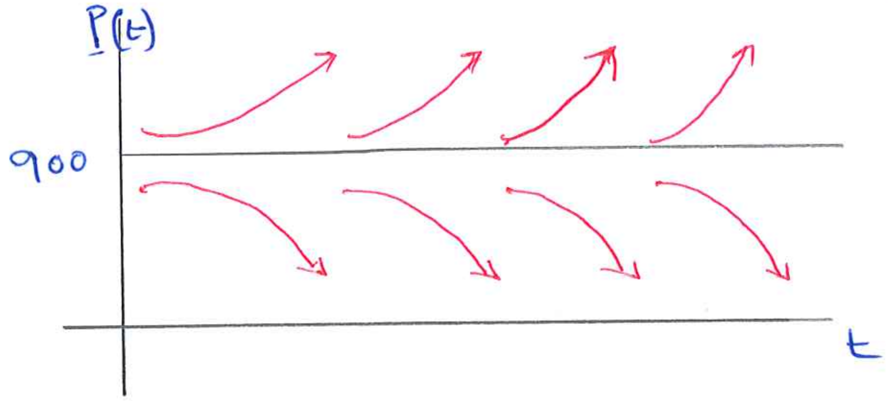
We will study the behavior of the solution of Eq. (4) without solving it.

Equilibrium solution : $\frac{dP}{dt} = 0$

$\Rightarrow \frac{1}{2}P - 450 = 0 \Rightarrow \boxed{P = 900}$

If $P_0 < 900$, (take $P_0 = 0$), then $\frac{dP}{dt} < 0$.

If $P_0 > 900$, (take $P_0 = 1000$), then $\frac{dP}{dt} > 0$



There fore:

$$\lim_{t \to \infty} p(t) = \begin{cases} +\infty &, \quad P_0 > 900 \\ \underbrace{0} &, \quad P_0 < 900 \end{cases}$$

Not $-\infty$, since $p(t)$ is a population.

(Q7) page 8. Write down a differential equation of the form $\frac{dy}{dt} = ay + b$, whose all solutions approach $y = 3$ as $t \longrightarrow \infty$.

Sol: For all solutions to approach the equilibrium solution $y(t) = 3$, we must have:

$y' < 0$, for $y > 3$, and $y' > 0$, for $y < 3$.

$\Rightarrow \quad y' = 0$ iff $y = 3$.

$\Rightarrow \quad y' = ay + b = 0 \iff y = -\frac{b}{a} = \frac{3}{1}$

Therefore: $a = 1$ & $b = -3$ (X)

$\therefore \quad y' = y - 3$. (X) (since in this case $y(t) \not\to 3$)

But, If we assume $a = -1$ and $b = 3$, then

$y' = -y + 3$. (✓)

(Q22) page 9. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Sol: Let V : Volume of the raindrop.

t : time.

S : surface area,

then $$\frac{dV}{dt} = -kS \quad , \text{ for } k > 0.$$

Since the Volume is given by $V = \frac{4}{3}\pi r^3$, ... (1)

where r: radius, then $S = 4\pi r^2$ ... (2)

Solve (1) for r, then $r = \left(\frac{3}{4\pi} V\right)^{\frac{1}{3}}$ ... (3)

Now, substitute (3) in (2), we get

$$S = 4\pi \left(\frac{3}{4\pi} V\right)^{2/3}$$

Thus: $$\frac{dV}{dt} = \underbrace{-k(4\pi)\left(\frac{3}{4\pi}\right)^{2/3}}_{C} V^{2/3} = -c\, V^{2/3}$$

with $c > 0$.

(Q23) page 9. Newton's law of Cooling: (Mentioned in 2.3 Also)

Newton's Law of Cooling states that the temprature of an object changes at a rate proportional to the difference between the temprature of the object and its surrounding temprature.

$$\left(\text{i-e}, \quad \frac{dT}{dt} = c\,(T - T_R)\right), \qquad \begin{pmatrix} c<0 \\ \text{cooling} \end{pmatrix} \quad \begin{pmatrix} c>0 \\ \text{heating} \end{pmatrix}$$

where T: temprature of the object

T_R: surrounding temprature. (Ambient temprature).

Suppose that the ambient temprature is $70^\circ F$ and the rate constant is $0.05\,(\text{min})^{-1}$. Write a differential equation for the temprature of the object at any time.

sol:

$$\frac{dT}{dt} = c\,(T - 70)$$

$$= -0.05\,(T - 70).$$

(cooling)

Note: If its Heating, we assume $c > 0$.

1.2 Solutions of Some Differential Equations:

Recall, in section 1.1, we derived the DE's

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v \quad \cdots (1) \quad (\text{Falling object}).$$

$$\frac{dp}{dt} = rp - k \quad \cdots (2) \quad (\text{population of field mice \& owls}).$$

Both DEs (1) and (2) are of the general form

$$\frac{dy}{dt} = ay - b. \quad , \quad a, b \in \mathbb{R}.$$

Our Aim is to find the exact solution of (1) and (2) for a given m, g, γ, r and k as follows:

Example (1): Solve $\frac{dp}{dt} = \frac{1}{2}p - 450$.

Sol: $\frac{dp}{dt} = \frac{p - 900}{2}$

$$\Longleftrightarrow \quad \frac{dp}{p-900} = \frac{1}{2} dt \quad , \quad p \neq 900. \quad \cdots (3)$$

Then by integrating both sides of (3), we get

$$\int \frac{dp}{p-900} = \int \frac{1}{2}\, dt$$

$$\Rightarrow \ln|p-900| = \frac{1}{2}t + C$$

$$\Rightarrow |p-900| = e^{\frac{1}{2}t + C} = e^{C} e^{\frac{1}{2}t}$$

$$\Rightarrow p - 900 = \pm e^{C} e^{\frac{1}{2}t}$$

$$\Rightarrow p(t) = 900 + A e^{\frac{1}{2}t}, \text{ where } A = \pm e^{C}.$$

(non Zero Constant)

Example (2): Solve the following IVP:

$$\begin{cases} \frac{dp}{dt} = \frac{1}{2}p - 450 \\ p(0) = 850 \end{cases}$$

We found: $p(t) = 900 + A e^{\frac{1}{2}t}$

Now : $p(0) = 900 + A = 850 \Rightarrow \boxed{A = -50}$

$$\therefore \quad p(t) = 900 - 50\, e^{\frac{1}{2}t}.$$

Notice that $\lim\limits_{t \to \infty} p(t) = 0$, (Not $-\infty$ because $p(t)$ is a population).

Example (3): Solve the following IVP:

$$\begin{cases} \dfrac{dV}{dt} = 9.8 - \frac{1}{5}V \\ V(0) = 0 \end{cases}$$

Sol: If $V \neq 49$, then:

$$\frac{dV}{9.8-\frac{1}{5}V} = dt \Rightarrow -5\int \frac{-\frac{1}{5}\,dV}{9.8-\frac{1}{5}V} = \int dt$$

$$\Rightarrow -5\ln\left|9.8-\frac{1}{5}V\right| = t + C_1.$$

$$\Rightarrow \ln\left|9.8-\frac{1}{5}V\right| = \frac{-t}{5} + C_2.$$

$$\Rightarrow \left|9.8-\frac{1}{5}V\right| = e^{C_2}\, e^{-t/5}$$

$$\Rightarrow 9.8-\frac{1}{5}V = \underbrace{\pm e^{C_2}}_{C_3} e^{-\frac{t}{5}} = C_3\, e^{-\frac{1}{5}t}$$

$$\Rightarrow \frac{1}{5}V = 9.8 - C_3\, e^{-\frac{1}{5}t}$$

$$\Rightarrow V(t) = 49 \underbrace{- 5C_3}_{B} e^{-\frac{1}{5}t} = 49 + B e^{-\frac{1}{5}t}$$

Now: $V(0) = 0$ implies $0 = 49 + B \Rightarrow \boxed{B = -49}$

$$\Rightarrow V(t) = 49 - 49\, e^{-\frac{1}{5}t}$$

Notice that, $\lim_{t\to\infty} V(t) = 49.$

Example (4): Solve the IVP:

$$\begin{cases} \dfrac{dy}{dt} = ay - b \\ y(0) = \alpha \end{cases}$$

If $y \neq \frac{b}{a}$, $a \neq 0$, we have:

$$\frac{1}{a}\int \frac{a\,dy}{ay-b} = \int dt \Rightarrow \frac{1}{a}\ln|ay-b| = t + C_1$$

$$\Rightarrow \ln|ay-b| = at + \underbrace{aC_1}_{C_2} = at + C_2$$

$$\Rightarrow |ay-b| = e^{C_2} e^{at}$$

$$\Rightarrow ay - b = \underbrace{\pm e^{C_2}}_{A} e^{at} = Ae^{at}$$

$$\Rightarrow ay(t) = Ae^{at} + b$$

$$\Rightarrow y(t) = \underbrace{\frac{A}{a}}_{B} e^{at} + \frac{b}{a} = \frac{b}{a} + Be^{at}$$

Using $y(0) = \alpha$, then $\alpha = \frac{b}{a} + B \Rightarrow \boxed{B = \alpha - \frac{b}{a}}$

Finally: $y(t) = \frac{b}{a} + \left(\alpha - \frac{b}{a}\right) e^{at}$.

Some Important Questions: ① Is there a Solution? (Existence)

② If the Solution exists, is it Unique? (Uniqueness).

③ How to find the Solution if it exists?

(Q3) page 16. Consider the differential equation

$$\frac{dy}{dt} = -ay + b, \quad \text{where } a, b > 0.$$

(a) Find the general Solution of the D.E.

(b) Sketch the Solution for several different initial Conditions.

(c) Describe how the Solutions change under each of the following Conditions:

i. a increases.

ii. b increases.

iii. Both a and b increase, but $\frac{b}{a}$ remains Constant

Sol: (a) $\dfrac{dy}{b-ay} = dt$. Integrate both sides:

provided $y \neq \frac{b}{a}$, then

$$-\frac{1}{a} \ln |b-ay| = t + c_1.$$

$$\Leftrightarrow \ln |b-ay| = -at - ac_1 = -at + C_2$$

$C_2 = -ac_1$

$$\Leftrightarrow \quad |b - ay| = e^{C_2} e^{-at}$$

$$\Leftrightarrow \quad b - ay = \pm e^{C_2} e^{-at} = C_3 e^{-at}, \quad C_3 = \pm e^{C_2}$$

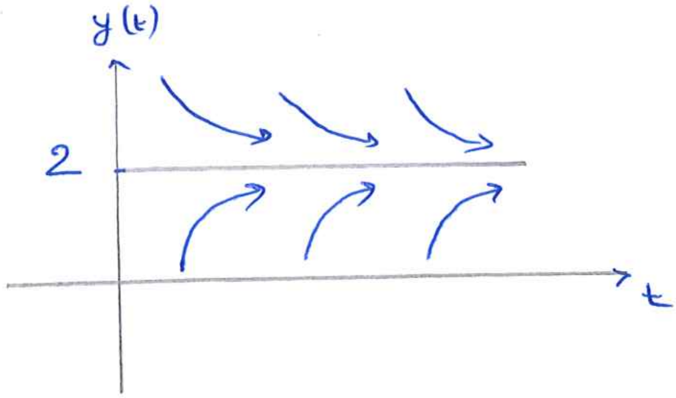
$$\Leftrightarrow \quad y = \frac{\left(b - C_3 e^{-at}\right)}{a}$$

(Note: If $y = \frac{b}{a}$, then $y' = 0$, and $y(t) = \frac{b}{a}$ is an Equilibrium Solution).

(b)

Assume $a = 1$

$b = 2$



(c)(i) If a increases, the Equilibrium Solution gets closer to $0 = y(t)$. (The Convergence rate increases).

(ii) If b increases, then Equilibrim $\overset{y(t)}{=} \frac{b}{a}$ becomes larger.

(The Convergence rate remains the same).

(iii) If a and b increase ($\frac{b}{a}$ constant), then the equilibrum $\overset{y(t)}{=} \frac{b}{a}$ remains the same. (Convergence rate Increase).

(Q9) page 17. The falling object satisfies the following IVP: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $v(0) = 0$.

(a) Find the time that must elapse for the object to reach 98 % of its limiting velocity.

(b) How far does the object fall in time found in part (a).

Sol: $v' = -\frac{1}{5}(v - 49)$

then $\frac{dv}{v-49} = -\frac{1}{5}\,dt$. Integrate both sides:

$$\ln|v-49| = -\frac{1}{5}t + c_1 \iff |v-49| = e^{c_1} e^{-\frac{1}{5}t}$$

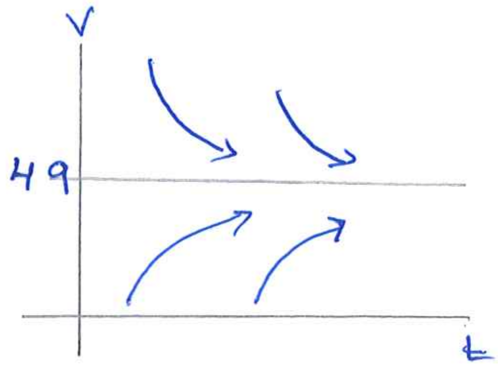
$$\iff v - 49 = \pm e^{c_1} e^{-\frac{1}{5}t} = c_2 e^{-\frac{1}{5}t}$$

$$\iff v = c_2 e^{-\frac{1}{5}t} + 49$$

Using $v(0) = 0$, then $0 = c_2 + 49 \Rightarrow c_2 = -49$

$$\therefore\ v(t) = 49 - 49e^{-\frac{1}{5}t} = 49\left(1 - e^{-\frac{1}{5}t}\right).$$

We note that $\lim_{t \to \infty} v(t) = 49$.



In order to find the time for which the object to reach 98% of its limiting velocity, set $v = 0.98(49)$, then

$$0.98(49) = 49\left(1 - e^{-\frac{1}{5}t}\right).$$

$$\Leftrightarrow \quad 0.98 = 1 - e^{-\frac{1}{5}t} \quad \Leftrightarrow \quad e^{-\frac{1}{5}t} = 0.02$$

$$\Leftrightarrow \quad -\frac{1}{5}t = \ln(0.02) \quad \Leftrightarrow \quad t = -5\ln(0.02).$$

$t \approx 19.56$ second.

(b) We know that $v(t) = \dfrac{dx}{dt}$. (x: position).

$$\Rightarrow x(t) = \int v(t)\,dt + C_3 = \int 49\left(1 - e^{-\frac{1}{5}t}\right)dt + C_3$$

$$\Rightarrow x(t) = 49t + 5(49)e^{-\frac{1}{5}t} + C_3$$

Use $x(0) = 0$: $\Rightarrow \quad 0 = 5(49) + C_3 \Rightarrow C_3 = -49(5)$

$$\Rightarrow x(t) = 49\left(t + 5e^{-\frac{1}{5}t}\right) - 49(5)$$

There fore: $x(-5\ln 0.02) = 49\left((-5\ln 0.02) + 0.02\right) - 49(5)$

≈ 718.3 meters
