

Chapter 3. Second Order Linear Equations.

3.1 Homogeneous Equations with Constant

Coefficients.

3.3 Complex roots of the characteristic Equation

A second order ODE has the form

$$\frac{d^2 y}{dt^2} = f(t, y, \frac{dy}{dt}) \quad \dots (1)$$

Eq. (1) is said to be linear if f has the

$$\text{form: } f(t, y, \frac{dy}{dt}) = g(t) - p(t) \frac{dy}{dt} - q(t) y \quad \dots (2)$$

i.e. f is linear in y and y'

So, eq (1) can be written as:

$$y'' + p(t) y' + q(t) y = g(t) \quad \dots (3)$$

If Eq (1) is not of the form of Eq. (3)

then it is called Nonlinear.

Note: ① If $g(t) = 0$ in Eq. (3), then we call it Homogeneous Linear Equation.

② If $g(t) \neq 0$ in Eq. (3), then we call it Nonhomogeneous Linear Equation.

In sections 3.1, 3.3 and a part of 3.4, we will seek for the solution of the following 2nd order Linear Homogeneous equation with Constant coefficients

$$a y'' + b y' + c y = 0 \quad \dots(4)$$

where a , b and c are constants.

To solve Eq. (4), we assume the solution

as $y = e^{rt} \Rightarrow \underline{y' = r e^{rt}}$ and $\underline{y'' = r^2 e^{rt}}$

substitute in Eq. (4) we get:

$$a y'' + b y' + c y = 0$$

$$\Leftrightarrow a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$$

$$\Leftrightarrow (a r^2 + b r + c) e^{rt} = 0$$

\Leftrightarrow $a r^2 + b r + c = 0$ which is called
the characteristic equation or auxiliary Eq.

Therefore, the roots of the Characteristic Eq. are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, we have 3 cases to be considered.

Case 1: r_1, r_2 are distinct real roots.

In this case: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

Case 2: $r_1 = r_2 = r$ (repeated roots) sec 3.4

In this case: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

Case 3: r_1, r_2 are conjugate complex roots

$$r_1 = \alpha + \beta i \quad , \quad r_2 = \alpha - \beta i \quad \text{sec. (3.3)}$$

In this case:

$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t.$$

Example: Solve the following D.Es

① $y'' + 3y' + 2y = 0$

The characteristic Equation is:

$$r^2 + 3r + 2 = 0$$

$$(r + 1)(r + 2) = 0$$

$$\Rightarrow r_1 = -1 \quad \text{and} \quad r_2 = -2$$

$$\Rightarrow y(t) = C_1 e^{-t} + C_2 e^{-2t}$$

② $y'' + 5y' + 6y = 0$

$$y(0) = 2 = y'(0).$$

The characteristic equation is:

$$r^2 + 5r + 6 = 0$$

$$\Rightarrow (r + 2)(r + 3) = 0$$

$$\Rightarrow r_1 = -2, r_2 = -3$$

$$\Rightarrow y(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

Now Using ICs:

$$y(0) = 2 \Rightarrow C_1 + C_2 = 2 \quad \dots \textcircled{+}$$

$$y'(t) = -2C_1 e^{-2t} + -3C_2 e^{-3t}$$

$$y'(0) = 2 \Rightarrow -2C_1 - 3C_2 = 2 \quad \dots \textcircled{+} \textcircled{+}$$

$$2 \text{ Eq } \textcircled{+} + \text{ Eq } \textcircled{+} \Rightarrow -C_2 = 6 \Rightarrow \boxed{C_2 = -6}$$

Substitute in either $\textcircled{+}$ or $\textcircled{+} \textcircled{+}$, we get

$$-2C_1 - 3(-6) = 2 \Rightarrow -2C_1 = -16 \Rightarrow \boxed{C_1 = 8}$$

$$\Rightarrow y(t) = 8 e^{-2t} + -6 e^{-3t}$$

$$\textcircled{3} \quad y'' + 6y' + 9y = 0$$

Charact. Eq. : $r^2 + 6r + 9 = 0$

$$(r + 3)(r + 3) = 0$$

$$\Rightarrow r_1 = r_2 = -3$$

$$\Rightarrow y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$\textcircled{4} \quad y'' + y' + 9.25y = 0, \quad y(0) = 0, \quad y'(0) = 9.$$

Charact. Eq. : $r^2 + r + 9.25 = 0$

$$\Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(9.25)}}{2(1)}$$

$$\Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{-36}}{2} \Rightarrow r_{1,2} = -\frac{1}{2} \pm 3i$$

$$\Rightarrow y(t) = c_1 e^{-\frac{1}{2}t} \cos 3t + c_2 e^{-\frac{1}{2}t} \sin 3t$$

Using I. C.s :

$$y(0) = 0 \Rightarrow \boxed{0 = c_1}$$

Since $c_1 = 0 \Rightarrow y(t) = c_2 e^{-\frac{1}{2}t} \sin 3t$.

$$y'(t) = 3c_2 e^{-\frac{1}{2}t} \cos 3t - \frac{1}{2}c_2 e^{-\frac{1}{2}t} \sin 3t.$$

$$\Rightarrow y'(0) = 9 \Rightarrow 9 = 3c_2 - 0 \Rightarrow \boxed{c_2 = 3}$$

$$\Rightarrow \underline{y(t) = 3 e^{-\frac{1}{2}t} \sin 3t.}$$

To discuss the Behaviour of $y(t)$:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} 3 e^{-\frac{1}{2}t} \sin 3t = 0$$

(Since $-1 \leq \sin 3t \leq 1$

$$\Rightarrow -3e^{-\frac{1}{2}t} \leq 3e^{-\frac{1}{2}t} \sin 3t \leq 3e^{-\frac{1}{2}t}$$

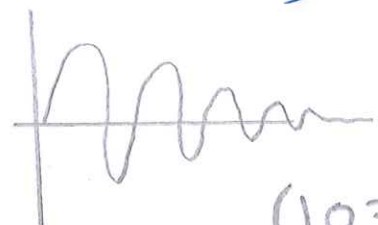
$$\lim_{t \rightarrow \infty} -3e^{-\frac{1}{2}t} \leq \lim_{t \rightarrow \infty} 3e^{-\frac{1}{2}t} \sin 3t \leq \lim_{t \rightarrow \infty} 3e^{-\frac{1}{2}t}$$

|| ||

0 0

Using Sandwich thm, $\lim_{t \rightarrow \infty} 3e^{-\frac{1}{2}t} \sin 3t = 0$).

(Decay Oscillation).



$$(5) \quad 16y'' - 8y' + 145y = 0$$

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1$$

The charact. Eq.

$$16r^2 - 8r + 145 = 0$$

$$\Rightarrow r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{2(16)}$$

$$= \frac{8 \pm \sqrt{64(1-145)}}{32} = \frac{8 \pm 8(12)i}{32}$$

$$r_{1,2} = \frac{1}{4} \pm 3i$$

$$\Rightarrow y(t) = c_1 e^{\frac{1}{4}t} \cos 3t + c_2 e^{\frac{1}{4}t} \sin 3t.$$

Using I.C: $y(0) = 0 \Rightarrow \boxed{0 = c_1}$

Therefore: $y(t) = c_2 e^{\frac{1}{4}t} \sin 3t$

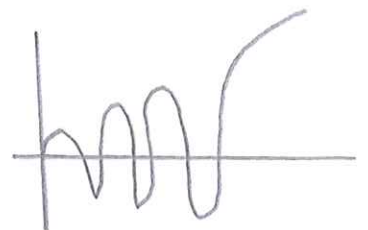
$$y'(t) = 3c_2 e^{\frac{1}{4}t} \cos 3t + \frac{1}{4}c_2 e^{\frac{1}{4}t} \sin 3t.$$

Using I.C: $y'(0) = 1 \Rightarrow 1 = 3c_2 \Rightarrow \boxed{c_2 = \frac{1}{3}}$

$$\Rightarrow y(t) = \frac{1}{3} e^{\frac{1}{4}t} \sin 3t.$$

Note that $\lim_{t \rightarrow \infty} y(t)$ is unbounded.

(Growing Oscillation).



Example: Let y be the solution of the IVP

$$y'' - y' - 2y = 0$$

$$y(0) = \alpha \quad \text{and} \quad y'(0) = 1$$

Find α for which $\lim_{t \rightarrow \infty} y(t) = 0$.

Sol: The characteristic equation is

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0 \Rightarrow r_1 = 2, r_2 = -1$$

$$\Rightarrow y(t) = c_1 e^{2t} + c_2 e^{-t}$$

Using I.C: $y(0) = \alpha \Rightarrow \boxed{c_1 + c_2 = \alpha} \dots \textcircled{1}$

$$y'(t) = 2c_1 e^{2t} - c_2 e^{-t}$$

Using I.C: $y'(0) = 1 \Rightarrow \boxed{2c_1 - c_2 = 1} \dots \textcircled{2}$

$$\therefore \textcircled{1} + \textcircled{2} \Rightarrow 3c_1 = 1 + \alpha \Rightarrow c_1 = \frac{1 + \alpha}{3}$$

$$\textcircled{1} \Rightarrow c_2 = \alpha - \frac{1 + \alpha}{3} = \frac{3\alpha - 1 - \alpha}{3} = \frac{2\alpha - 1}{3}$$

$$\therefore y(t) = \left(\frac{1 + \alpha}{3}\right) e^{2t} + \left(\frac{2\alpha - 1}{3}\right) e^{-t}$$

Since $\lim_{t \rightarrow \infty} y(t) = 0$ and $e^{2t} \xrightarrow{t \rightarrow \infty} \infty$, then:

$$\frac{\alpha + 1}{3} = 0 \iff \boxed{\alpha = -1}$$

Home work #9:

1) Let $y'' + y' - 2y = 0$, $y(0) = \beta$ and $y'(0) = 2$

Find the values of β for which $\lim_{t \rightarrow \infty} y(t) = 0$

2) Consider $y'' + 2\alpha y' + y = 0$

Assume that the characteristic equation has

Complex roots. Find the value of α for which

$$\lim_{t \rightarrow \infty} y(t) = 0$$

Euler's formula:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

→ Proof
Using
Maclaurin
Series.

Example: Use Euler's formula to write the given expression in the form $a + bi$

$$\begin{aligned} \textcircled{1} \quad e^{3i\pi} &= e^{i(3\pi)} = \cos(3\pi) + i \sin(3\pi) \\ &= -1 + i(0) = -1 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad e^{2 + \frac{\pi}{2}i} &= e^2 \cdot e^{\frac{\pi}{2}i} = e^2 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right) \\ &= e^2 (0 + i) = e^2 i. \end{aligned}$$

$$\textcircled{3} \quad \pi^{-1+2i} = \frac{(-1+2i) \ln \pi}{e} = e^{-\ln \pi} \cdot e^{i(2 \ln \pi)}$$

$$= \frac{1}{\pi} \left(\cos(2 \ln \pi) + i \sin(2 \ln \pi) \right)$$

Example: Show that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Sol: $\frac{e^{ix} - e^{-ix}}{2i} = \frac{(\overset{\text{even}}{\cos x} + i \overset{\text{odd}}{\sin x}) - (\cos(-x) + i \sin(-x))}{2i}$

$$= \frac{\cancel{\cos x} + i \sin x - \cancel{\cos x} + i \sin x}{2i} = \sin x$$

Note: In Case (3), complex roots $r_{1,2} = \alpha + \beta i$

the solution is given by:

$$y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

Since: $y_1(t) = e^{r_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$

$$y_2(t) = e^{r_2 t} = e^{(\alpha - i\beta)t} = e^{\alpha t} (\cos(-\beta t) + i \sin(-\beta t))$$

$$\Rightarrow y(t) = c_1 y_1(t) + c_2 y_2(t) =$$

$$= c_1 e^{\alpha t} (\cos \beta t + i \sin \beta t) + c_2 e^{\alpha t} (\cos \beta t - i \sin \beta t)$$

$$= \underbrace{(c_1 + c_2)}_D e^{\alpha t} \cos \beta t + \underbrace{(c_1 i - c_2 i)}_C e^{\alpha t} \sin \beta t$$

$$= D e^{\alpha t} \cos \beta t + C e^{\alpha t} \sin \beta t$$

(Q 20) page 144, Find the solution of IVP

$$2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

sol: The charact. Eq: $2r^2 - 3r + 1 = 0$

$$(2r - 1)(r - 1) = 0$$

Therefore $r_1 = \frac{1}{2}$, $r_2 = 1$

$$\Rightarrow y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^t$$

Using I.C: $y(0) = 2 \Rightarrow 2 = c_1 + c_2 \dots \textcircled{1}$

$$y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} + c_2 e^t$$

Using I.C: $y'(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2} c_1 + c_2 \dots \textcircled{2}$

$$\Rightarrow \boxed{c_1 = 3} \text{ and } \boxed{c_2 = -1}$$

$$\Rightarrow y(t) = 3e^{\frac{1}{2}t} - e^t.$$

Maximum occurs at $y' = 0$ & $y'' < 0$

$$y'(t) = \frac{3}{2} e^{\frac{1}{2}t} - e^t = 0 \Rightarrow \boxed{t = \ln \frac{9}{4}}$$

Note that $y''(\ln \frac{9}{4}) < 0$.

$$y(t) = 0 \Leftrightarrow 3e^{\frac{1}{2}t} - e^t = 0 \Rightarrow \boxed{t = \ln 9}$$

(Q 34) page 166 (section 3.3)

Euler Equations: (Non Constant Coefficients).

An equation of the form :

$$A t^2 y'' + B t y' + C y = 0, \quad t > 0 \quad \dots (*)$$

where $A, B, C \in \mathbb{R}$ is called Euler Equation.

To solve it let $x = \ln t$ or $t = e^x$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \left(\frac{1}{t}\right) \quad \dots (1)$$

$$\frac{d^2 y}{dt^2} = \frac{dy}{dx} \left(-\frac{1}{t^2}\right) + \frac{1}{t} \cdot \frac{d}{dt} \left(\frac{dy}{dx}\right)$$

$$= -\frac{1}{t^2} \frac{dy}{dx} + \frac{1}{t} \frac{d^2 y}{dx^2} \cdot \underbrace{\left(\frac{dx}{dt}\right)}_{\frac{1}{t}} \quad \dots (2)$$

منه الداخل

Now substitute (1) & (2) in Eq. (*):

$$A t^2 \left(-\frac{1}{t^2} \frac{dy}{dx} + \frac{1}{t^2} \frac{d^2 y}{dx^2} \right) + B t \left(\frac{1}{t} \frac{dy}{dx} \right) + C y = 0$$

$$\Rightarrow -A \frac{dy}{dx} + A \frac{d^2 y}{dx^2} + B \frac{dy}{dx} + C y = 0$$

$$\Rightarrow A \frac{d^2 y}{dx^2} + (B - A) \frac{dy}{dx} + C y = 0. \quad \dots (**)$$

(109)

Notice that eq. (**) is a 2nd order Linear Homogeneous with constant coefficients.

Therefore, the characteristic equation is:

$$Ar^2 + (B-A)r + C = 0$$

with
$$r_{1,2} = \frac{-(B-A) \pm \sqrt{(B-A)^2 - 4AC}}{2A}$$

So:

1) If $r_1 \neq r_2 \in \mathbb{R}$, then:

$$y(\underline{x}) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \xrightarrow{y(t)} c_1 e^{r_1 \ln t} + c_2 e^{r_2 \ln t}$$

$$y(\underline{t}) = c_1 t^{r_1} + c_2 t^{r_2}.$$

2) If $r_1 = r_2 = r$, then

$$y(\underline{x}) = c_1 e^{rx} + c_2 x e^{rx} = c_1 e^{r \ln t} + c_2 \ln t e^{r \ln t}$$

$$y(\underline{t}) = c_1 t^r + c_2 (\ln t) t^r.$$

3) If $r_{1,2} = \alpha + \beta i$, then

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$y(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t). \quad (110)$$

(Q36) page 166. Solve the following D. E.

$$t^2 y'' + 4t y' + 2y = 0, \quad t > 0$$

Let $x = \ln t$, using Eq. (**) with

$A = 1$, $B = 4$, $C = 2$, we have:

$$\frac{d^2 y}{dx^2} + (4-1) \frac{dy}{dx} + 2y = 0$$

$$\Leftrightarrow \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = 0$$

Characteristic Eq. : $r^2 + 3r + 2 = 0$

$$(r+1)(r+2) = 0$$

$$\Rightarrow r = -1 \quad \text{and} \quad r = -2$$

$$\Rightarrow y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

$$\Rightarrow y(t) = c_1 e^{-\ln t} + c_2 e^{-2 \ln t}$$

$$= c_1 t^{-1} + c_2 t^{-2}$$

$$= \frac{c_1}{t} + \frac{c_2}{t^2}$$

(III)

Example: Solve $4t^2 y'' + 12ty' + 5y = 0, t > 0$

So 1: Let $x = \ln t$, then Using Eq. (**):

with $A = 4, B = 12, C = 5$:

$$4 \frac{d^2 y}{dx^2} + (12 - 4) \frac{dy}{dx} + 5y = 0$$

$$\Rightarrow 4 \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 5y = 0.$$

The charact. Eq. : $4r^2 + 8r + 5 = 0$

$$\Rightarrow r_{1,2} = \frac{-8 \pm \sqrt{64 - 4(5)(4)}}{2(4)}$$

$$= \frac{-8 \pm 4i}{8} = -1 \pm \frac{1}{2}i$$

$$\Rightarrow y(x) = C_1 e^{-x} \cos\left(\frac{1}{2}x\right) + C_2 e^{-x} \sin\left(\frac{1}{2}x\right)$$

$$= C_1 e^{-\ln t} \cos\left(\frac{1}{2} \ln t\right) + C_2 e^{-\ln t} \sin\left(\frac{1}{2} \ln t\right)$$

$$= \frac{C_1}{t} \cos\left(\frac{1}{2} \ln t\right) + \frac{C_2}{t} \sin\left(\frac{1}{2} \ln t\right).$$

3.2 Solutions of Linear Homogeneous Equations ;

The Wronskian.

Theorem (3.2.1): (Existence and Uniqueness Thm):

$$\text{Consider the IVP : } \begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y'_0 \end{cases}$$

where p, q and g are continuous on an open interval $I = (\alpha, \beta)$ ^{that} contains t_0 , then there is exactly one solution for the IVP.

Example: Find the largest interval in which the solution of the IVP is certain to exist.

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0$$

$$y(1) = 2, \quad y'(1) = 1$$

Sol:
$$y'' + \frac{t}{t(t-3)}y' - \frac{(t+3)}{t(t-3)}y = 0$$

$$p(t) = \frac{t}{t(t-3)}, \quad q(t) = -\frac{(t+3)}{t(t-3)}, \quad g(t) = 0$$

p , q and g are continuous on $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$

So, the largest interval that contains $t_0 = 1$ is

$(0, 3)$ in which the solution is certain to exist.

Linear operator:

$$\text{Define } L[y] = y'' + p(t)y' + q(t)y = 0 \quad \dots \textcircled{1}$$

If y_1 is a solution of the D.E., then $L[y_1] = 0$.

$$(i-c) \quad L[y_1] = y_1'' + p(t)y_1' + q(t)y_1 = 0$$

Theorem (3.2.2): (principle of Superposition).

If y_1 and y_2 are two solutions of Eq (1),

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any values of c_1 and c_2 .

Proof: Suppose $L[y_1] = 0$ and $L[y_2] = 0$
need to show $L[c_1 y_1 + c_2 y_2] = 0$?

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= (c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2) \\ &= c_1 y_1'' + c_2 y_2'' + p(t)(c_1 y_1') + p(t)(c_2 y_2') + q(t)c_1 y_1 + q(t)c_2 y_2 \\ &= c_1 (y_1'' + p(t)y_1' + q(t)y_1) + c_2 (y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1 L[y_1] + c_2 L[y_2] = c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

$\Rightarrow L[c_1 y_1 + c_2 y_2] = 0$ (i.e) $c_1 y_1 + c_2 y_2$ is a solution

Note: We call $c_1 y_1 + c_2 y_2$ a General solution.

Def: The Wronskian of the solutions y_1 and y_2

is given by: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Example: Find $W(t, t \ln t)$, $t > 0$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t \ln t \\ 1 & 1 + \ln t \end{vmatrix}$$
$$= t(1 + \ln t) - t \ln t = t$$

Theorem (3.2.3) Suppose that y_1 and y_2 are two solutions of the IVP:

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \quad (2)$$

Then by thm (3.2.2), it is always possible to choose the constants c_1 and c_2 so that $c_1 y_1 + c_2 y_2$ satisfies the D.E. iff $W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$ at $t_0 \in I$.

Proof: By thm (3.2.2), we can find c_1 and c_2 such that $y(t) = c_1 y_1 + c_2 y_2$ is a solution.

$$\text{Using I.C : } y(t_0) = y_0 \Rightarrow \left\{ \begin{array}{l} y_0 = c_1 y_1(t_0) + c_2 y_2(t_0) \\ \text{and } y'(t_0) = y_0' \Rightarrow y_0' = c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{array} \right\}$$

↓

Solve the System (116)

Using Cramer's Method, we can find C_1 and C_2 .

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \quad \text{and} \quad C_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

So, for C_1 and C_2 to make sense, we should have

$$\underline{W(y_1, y_2)(t_0) \neq 0.}$$

Theorem (3.2.4): Suppose that y_1 and y_2 are two solutions of the D.E.s.

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad \dots (3)$$

Then the family of solutions

$$y = C_1 y_1 + C_2 y_2, \quad C_1, C_2 \text{ arbitrary.}$$

includes all solutions of Eq (3) if and only if

there is a point t_0 where $\underline{W(y_1, y_2)(t_0) \neq 0.}$

Def: We say that y_1 and y_2 are Linearly independent on I , if $c_1 y_1 + c_2 y_2 = 0$

Implies $c_1 = c_2 = 0$

Example: $f(x) = \sin 2x$, $g(x) = \sin x \cos x$

If $c_1 \sin 2x + c_2 \sin x \cos x = 0$, then

We can find $c_1 = 1$ and $c_2 = -2$ such that

$$1(\sin 2x) - 2 \sin x \cos x = 0$$

then $f(x)$ and $g(x)$ are Linearly dependent.

Def: We say that y_1 and y_2 are Linearly

Independent on I iff $W(y_1, y_2)(t) \neq 0$, for

at least one $t \in I$.

Example: $y_1 = e^{2t}$ and $y_2 = e^{3t}$

are L.I

on $(-\infty, \infty)$ since:

Linearly Independent

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 3e^{5t} - 2e^{5t} = e^{5t} \neq 0, \forall t \in \mathbb{R}$$

Example: Are $\{1, x, x^2\}$ Linearly Independent?

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

$\therefore \{1, x, x^2\}$ are Linearly independent on \mathbb{R} .

Remark: If y_1 and y_2 are Linearly dependent,

then $W(y_1, y_2) = 0$.

But the converse is Not true, (i.e) if

$W(y_1, y_2) = 0$, we don't know if y_1, y_2 are

Linearly Independent or not, we need to check!

Def: (Fundamental set of Solutions)

The solutions y_1 and y_2 are said to form a

Fundamental set of solutions, (FSS), of the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

iff $W(y_1, y_2) \neq 0$.

Example: Show that $y_1 = \sqrt{t}$ and $y_2 = \frac{1}{t}$ form a fundamental set of solution for

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0$$

Sol: (1) $y_1 = \sqrt{t}$ is a solution of the D.E, since

$$\begin{aligned} & 2t^2 \left(\frac{-1}{4t^{3/2}} \right) + 3t \left(\frac{1}{2\sqrt{t}} \right) - \sqrt{t} \\ &= -\frac{1}{2} t^{2-3/2} + \frac{3}{2} t^{1-1/2} - \sqrt{t} \\ &= -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0 \end{aligned}$$

Similarly, $y_2 = \frac{1}{t}$ is also a solution, since:

$$2t^2 \left(\frac{2}{t^3} \right) + 3t \left(\frac{-1}{t^2} \right) - \frac{1}{t} = 0.$$

$$(2) \quad W(\sqrt{t}, \frac{1}{t}) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2} \cdot \frac{1}{\sqrt{t^3}} \neq 0$$

$\Rightarrow \{ \sqrt{t}, \frac{1}{t} \}$ are Linearly Independent.

Therefore $\{ \sqrt{t}, \frac{1}{t} \}$ Form a F.S.S.

Example: Find FSS of $3y'' + y' - 2y = 0$.

Sol: characteristic eqn. $3r^2 + r - 2 = 0$

$$\Rightarrow (3r - 2)(r + 1) = 0 \Rightarrow r_1 = \frac{2}{3} \text{ \& } r_2 = -1$$

$$\therefore y_1 = e^{\frac{2}{3}t} \quad , \quad y_2 = e^{-t}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{\frac{2}{3}t} & e^{-t} \\ \frac{2}{3}e^{\frac{2}{3}t} & -e^{-t} \end{vmatrix} = -\frac{5}{3}e^{-\frac{1}{3}t} \neq 0, \forall t \in \mathbb{R}$$

$\Rightarrow \{ e^{\frac{2}{3}t}, e^{-t} \}$ Form a FSS.

Home work #10

(1) Check if $y_1 = x$ and $y_2 = xe^x$ form a FSS for the D.E

$$x^2 y'' - x(x+2)y' + (x+2)y = 0, x > 0$$

(2) Check if $y_1 = x$, $y_2 = \sin x$ form a FSS for the D.E

$$(1 - x \cot x) y'' - x y' + y = 0, 0 < x < \pi.$$

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Theorem (3.2.5): Consider the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on some open interval I . Choose a point $t_0 \in I$.

Let y_1 be the solution of $L[y] = 0$ and satisfies

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0, \quad \text{and let}$$

y_2 be another solution of $L[y] = 0$ and satisfies

$$y_2(t_0) = 0, \quad y_2'(t_0) = 1, \quad \text{then}$$

y_1 and y_2 form a fundamental set of solutions of $L[y] = 0$.

Remark: $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Hence by this theorem & thm (3.2.1) we observe that

the existence of the functions y_1 and y_2 is ensured

and that they form a fundamental set of solutions.

Example: Find the FSS specified by thm 3.2.5

for $y'' - y = 0$ using $t_0 = 0$.

Sol: Charac. equation: $r^2 - 1 = 0 \Leftrightarrow r_1 = 1$ & $r_2 = -1$

$$\therefore y_1 = e^t \text{ and } y_2 = e^{-t}$$

$$\therefore y(t) = c_1 e^t + c_2 e^{-t}$$

$$\text{Using } \left. \begin{array}{l} y(0) = 1 \Rightarrow c_1 + c_2 = 1 \\ y'(0) = 0 \Rightarrow c_1 - c_2 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{array}$$

$$\therefore y_3 = \frac{1}{2} e^t + \frac{1}{2} e^{-t} = \cosh t.$$

$$\text{Now: } \left. \begin{array}{l} y(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ y'(0) = 1 \Rightarrow c_1 - c_2 = 1 \end{array} \right\} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array}$$

$$\therefore y_4 = \frac{1}{2} e^t - \frac{1}{2} e^{-t} = \frac{e^t - e^{-t}}{2} = \sinh t$$

$\therefore \{y_3, y_4\} = \{\cosh t, \sinh t\}$ form a Fundamental

set of solutions.

Theorem (3.2.6) (Abel's theorem):

If y_1 and y_2 are two solutions of the d.e.

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I

then $W(y_1, y_2) = C e^{-\int p(t) dt}$, where C is

constant that depends on y_1 and y_2 , but not on t .

Moreover, if $C = 0$, then $W(y_1, y_2) = 0, \forall t \in I$.

If $C \neq 0$, $W(y_1, y_2) \neq 0, \forall t \in I$.

Proof: Since y_1 and y_2 are solutions of $L[y] = 0$

then $y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad \dots (1)$

and $y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad \dots (2)$

Multiply Eq. (1) by $-y_2$, and Eq. (2) by $+y_1$

then add the resulting equations, we obtain:

$$\textcircled{+} \begin{cases} -y_2 y_1'' + p(t) y_2 y_1' - q(t) y_1 y_2 = 0 \\ y_1 y_2'' + p(t) y_1 y_2' + q(t) y_1 y_2 = 0 \end{cases}$$

$$\underbrace{y_1 y_2'' - y_2 y_1''}_{W'} + p(t) \underbrace{(y_1 y_2' - y_2 y_1')} = 0 \quad \dots (3)$$

Notice that $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (y_1 y_2' - y_2 y_1')$

$$W'(y_1, y_2) = y_1 y_2'' + \cancel{y_1' y_2'} - y_2 y_1'' - \cancel{y_2' y_1'}$$

$$W' = y_1 y_2'' - y_2 y_1''$$

\therefore Eq. (3) becomes: $W' + p(t) W = 0$, (separable).

$$\Rightarrow \frac{dW}{W} = -p(t) dt \Rightarrow \int \frac{dW}{W} = -\int p(t) dt$$

$$\Rightarrow \ln |W| = -\int p(t) dt + C_1$$

$$\Rightarrow W = \pm e^{C_1} e^{-\int p(t) dt}$$

$$\Rightarrow W = C e^{-\int p(t) dt}, \text{ where } C \text{ is Constant.}$$

Since $e^{-\int p(t) dt} \neq 0$, then $W(y_1, y_2) \neq 0$ unless $C = 0$.

Example: Find the Wronskian of two solutions

$$of \quad t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0$$

Sol: $y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 0.$

$$P(t) = -\left(\frac{t+2}{t}\right) = -1 - \frac{2}{t}$$

$$\begin{aligned} \therefore W(t) &= C e^{-\int p(t) dt} = C e^{\int 1 + \frac{2}{t} dt} = C e^{t + 2 \ln t} \\ &= C t^2 e^t. \quad t > 0. \end{aligned}$$

Example: Suppose that $y_1 = \sqrt{t}$ is a solution of, $t > 0$

$$2t^2 y'' + 3t y' - y = 0. \quad \text{Find 2nd Independent Solution.}$$

Sol: $W(y_1, y_2) = \begin{vmatrix} \sqrt{t} & y_2 \\ \frac{1}{2\sqrt{t}} & y_2' \end{vmatrix} = \sqrt{t} y_2' - \frac{y_2}{2\sqrt{t}}.$

And $W(y_1, y_2) = C e^{-\int p(t) dt} = C e^{-\int \frac{3}{2} \cdot \frac{1}{t} dt}$

$$\therefore W(y_1, y_2) = C e^{-\frac{3}{2} \ln t} = C t^{-\frac{3}{2}} = \frac{C}{t^{3/2}}.$$

$$\therefore \sqrt{t} y_2' - \frac{1}{2\sqrt{t}} y_2 = \frac{C}{t^{3/2}}$$

$$\Rightarrow y_2' - \frac{1}{2t} y_2 = \frac{C}{t^2}, \quad \begin{array}{l} \text{(1st Order)} \\ \text{Linear} \end{array}$$

Using Integrating factor :

$$\mu(t) = e^{\int \frac{-1}{2t} dt} = e^{-\frac{1}{2} \ln t} = t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}}$$

$$\therefore y_2(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) dt + C' \right]$$

$$= \sqrt{t} \left[\int \frac{1}{\sqrt{t}} \cdot \frac{C}{t^2} dt + C' \right]$$

$$= \sqrt{t} \left[C \left(\frac{-2}{3} \right) t^{-\frac{3}{2}} + C' \right]$$

$$= \sqrt{t} \left[A t^{-\frac{3}{2}} + C' \right]$$

$$y_2(t) = A t^{-1} + C' \sqrt{t}.$$

Remark: Using Abel's theorem, we can find the

Wronskian of any two solutions of the differential

equation, without knowing the two solutions.

Thm: Consider the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \text{ where}$$

p and q are continuous functions.

If $y = u(t) + i v(t)$ is a complex solution of the D.E., then $u(t)$ and $v(t)$ are two solutions of the D.E.

Proof: $L[y] = L[u(t) + i v(t)] = 0$ (Complex solution)

$$= [u''(t) + i v''(t)] + p(t)[u'(t) + i v'(t)] + q(t)[u(t) + i v(t)]$$

$$= [u''(t) + p(t)u'(t) + q(t)u(t)] + i [v''(t) + p(t)v'(t) + q(t)v(t)]$$

$$= L[u(t)] + i L[v(t)] = 0$$

$$\Leftrightarrow L[u(t)] = L[v(t)] = 0$$

Q34) (page 157). If $ty'' + 2y' + te^t y = 0$

has y_1 and y_2 as a fundamental set of solutions

and if $W(y_1, y_2)(1) = 2$. Find $W(y_1, y_2)(5)$.

Sol: $y'' + \frac{2}{t} y' + e^t y = 0$

$$W(y_1, y_2) = C e^{-\int \frac{2}{t} dt} = C e^{-2 \ln|t|} = C(t)^{-2}$$

$$\text{Now, } W(y_1, y_2)(1) = 2 \Rightarrow 2 = C(1)^{-2}$$

$$\Rightarrow \boxed{C=2}$$

$$\Rightarrow W(y_1, y_2)(t) = 2(t^{-2})$$

$$\Rightarrow W(y_1, y_2)(5) = 2(5)^{-2} = \frac{2}{25}$$

3.4 Repeated Roots, Reduction of Order.

Reduction of Order Method.

Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0 \quad \dots (*)$$

Suppose we know one solution $y_1(t)$ of Eq. (*).

To find a second ^{L.I.} solution for Eq. (*), we let

$$y_2(t) = v(t)y_1(t) \quad \dots (I)$$

then

$$y_2'(t) = v(t)y_1'(t) + v'(t)y_1(t)$$

$$y_2''(t) = v(t)y_1''(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v''(t)y_1(t)$$

$$y_2''(t) = v(t)y_1''(t) + 2v'(t)y_1'(t) + v''(t)y_1(t)$$

Substitute y_2 , y_2' & y_2'' in Eq. (*) we get:

$$[v y_1'' + 2v' y_1' + v'' y_1] + p(t)[v y_1' + v' y_1] + q(t)[v y_1] = 0$$

$$\Rightarrow v [y_1'' + p(t)y_1' + q(t)y_1] + v' [2y_1' + p(t)y_1] + v'' y_1 = 0$$

0, since y_1 is a solution of Eq. (*)

$$\Rightarrow y_1 v'' + [2y_1' + p(t)y_1] v' = 0 \quad \dots (**)$$

Let $w = v'$, then $w' = v''$, then (**) becomes

$$y_1 w' + (2y_1' + p(t)y_1) w = 0 \quad \dots (***)$$

Eq. (***) is 1st order linear ODE.

$$(***) \Rightarrow w' + \left(\frac{2y_1'}{y_1} + p(t) \right) w = 0, \quad y_1 \neq 0.$$

$$M(t) = e^{\int \left(\frac{2y_1'}{y_1} + p(t) \right) dt} = e^{2 \ln|y_1| + \int p(t) dt}$$

$$= y_1^2 e^{\int p(t) dt}$$

$$\therefore w(t) = \frac{1}{y_1^2} e^{-\int p(t) dt} \left[\int 0 \cdot (y_1^2 e^{\int p(t) dt}) dt + C \right]$$

$$\therefore w(t) = \frac{C e^{-\int p(t) dt}}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$$

$$\therefore v'(t) = \frac{W(y_1, y_2)}{y_1^2}$$

From Eq. (I):

$$\therefore v(t) = \frac{y_2}{y_1} = \int \frac{W(y_1, y_2)}{y_1^2} dt.$$

$$\Rightarrow y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2} dt. \quad (\text{Reduction Formula})$$

Example: Given that $y_1 = t^2$ is a solution of

$$t y'' - 6 y' + \frac{10}{t} y = 0, \quad t > 0.$$

Use the method of Reduction of order to find a second Linearly Independent solution of the given D.E.

Sol: Let $y_2 = v y_1 = t^2 v(t)$

$$y_2' = t^2 v'(t) + 2t v(t)$$

$$y_2'' = t^2 v''(t) + 2t v'(t) + 2t v'(t) + 2v(t)$$

$$y_2'' = t^2 v''(t) + 4t v'(t) + 2v(t)$$

Substitute y_2 , y_2' and y_2'' in the D.E, we get:

$$t \left(t^2 v''(t) + 4t v'(t) + 2v(t) \right) - 6 \left(t^2 v'(t) + 2t v(t) \right) + \frac{10}{t} \left(t^2 v(t) \right)$$

$$\Rightarrow t^3 v''(t) + \underbrace{4t^2 v'(t)} + 2t v(t) - \underbrace{6t^2 v'(t)} - 12t v(t) + 10t v(t)$$

$$\Rightarrow t^3 v''(t) - 2t^2 v'(t) = 0$$

$$\Rightarrow v''(t) - \frac{2}{t} v'(t) = 0, \quad t > 0.$$

Let $w = v'$, then $w' = v''$,

$$\Rightarrow w' - \frac{2}{t}w = 0, \quad t > 0$$

$$M(t) = e^{\int -\frac{2}{t} dt} = e^{-2 \ln t} = t^{-2}, \quad t > 0$$

$$\Rightarrow w(t) = \frac{1}{t^{-2}} \left[\int t^{-2} (0) dt + C \right]$$

$$w(t) = Ct^2$$

$$\Rightarrow v'(t) = Ct^2 \Rightarrow v(t) = \frac{Ct^3}{3} + D$$

$$\therefore v(t) = At^3 + D, \quad \text{where } A = \frac{C}{3}.$$

$$\therefore y_2(t) = (At^3 + D)y_1 = (At^3 + D)t^2$$

$$\therefore y_2(t) = At^5 + Dt^2$$

$$\therefore y_2(t) = At^5$$

Example: Use the Reduction Formula to find y_2

in the previous example.

sol: $y'' - \frac{6}{t} y' + \frac{10}{t^2} y = 0, t > 0.$

$$W(y_1, y_2) = C e^{-\int \frac{6}{t} dt} = C e^{6 \ln|t|} = C t^6, t > 0.$$

$$\therefore y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2} dt = t^2 \int \frac{C t^6}{t^4} dt$$

$$y_2 = t^2 \left(C \frac{t^3}{3} \right) = \frac{C}{3} t^5 = A t^5.$$

Home Work #11.

(1) Given that $y_1 = \frac{1}{t}$ is a solution of
 $2t^2 y'' + 3t y' - y = 0, t > 0.$

Use the Method of Reduction of order to find y_2 ?

(2) Given that $y_1 = \frac{\sin x}{\sqrt{x}}$ is one solution of
 $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0, x > 0.$

Find y_2 Using the Reduction Formula.

3.5 Nonhomogeneous Equations, Method of undetermined Coefficients.

Consider the nonhomogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad \dots (1)$$

where p , q and g are continuous functions on an open interval I . The corresponding homogeneous differential equation of Eq. (1) is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad \dots (2)$$

Theorem 3.5.1: ① If Y_1 and Y_2 are two solutions of Eq. (1),

then $Y_1 - Y_2$ is a solution of Eq. (2)

② If y_1, y_2 are fundamental set of solutions of

Eq. (2), then $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.

proof: ① since Y_1 and Y_2 are two solutions of

Eq. (1), then: $L[Y_1] = g(t)$ and $L[Y_2] = g(t)$

$$\Rightarrow L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

Therefore $Y_1 - Y_2$ is a solution of Eq. (2).

② Since $Y_1 - Y_2$ is a solution of Eq. (2), and

y_1 and y_2 are fundamental set of solutions of Eq. (2),

then $Y_1 - Y_2$ can be written as a linear combination

$$\text{of } y_1 \text{ and } y_2, \text{ (i.e.) } \underline{Y_1 - Y_2 = c_1 y_1 + c_2 y_2}$$

Example: prove that if Y_1, Y_2 are solutions of

$L[y] = g(t)$, then $\frac{1}{4} Y_1 + \frac{3}{4} Y_2$ is also a solution

of $L[y] = g(t)$.

Proof: $L\left[\frac{1}{4} Y_1 + \frac{3}{4} Y_2\right] = \frac{1}{4} \underbrace{L[Y_1]} + \frac{3}{4} \underbrace{L[Y_2]}$

$$= \frac{1}{4} g(t) + \frac{3}{4} g(t)$$

$$= g(t)$$

Y_1 & Y_2 are two
solutions of

$$L[y] = g(t).$$

Method of Undetermined Coefficients'

Consider the nonhomogeneous 2nd order Linear D.E

$$a y'' + b y' + c y = g(t) \quad \dots (3).$$

where a, b, c are constants, and $g(t)$

is one of the following:

1) $g(t)$ is a polynomial

2) $g(t)$ is an exponential function (ex. e^{rt}).

3) $g(t)$ is sin or cosine functions, ($\sin \beta t$ or $\cos \beta t$)

4) $g(t)$ is a finite sums and products of (1), (2) and (3).

Remark: This method is limited to Linear D.E (3)

where the conditions on a, b, c and $g(t)$ as above.

Now, to solve Eq. (3) by this method, we must do the following:

1. Find y_h = the solution of the corresponding homogeneous equation: $a y'' + b y' + c y = 0$

2. Find a particular solution y_p of the nonhomogeneous

Eq. (3). Note that y_p depends totally on the

form of $g(t)$ as follows:

i) If $q(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$, poly.

then we let $y_p = A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$.

We substitute y_p in Eq.(3) to find A_0, A_1, \dots, A_n .

Then check that y_p is linearly independent to y_h .

If Not, we multiply by t^s with $s=0, 1$ or 2 .

Example: Find y_p for $y'' - y' - 6y = 1 - 3x^2$.

$$y_h : r^2 - r - 6 = 0 \Leftrightarrow (r+2)(r-3) = 0 \Leftrightarrow r = -2, 3$$

$$\therefore y_h(x) = c_1 e^{3x} + c_2 e^{-2x}$$

$$\text{Now: Let } y_p = A_0 + A_1 x + A_2 x^2$$

$$y_p' = A_1 + 2A_2 x$$

$$y_p'' = 2A_2$$

$$\Rightarrow 2A_2 - (A_1 + 2A_2 x) - 6(A_0 + A_1 x + A_2 x^2) = 1 - 3x^2$$

$$\Rightarrow (2A_2 - A_1 - 6A_0) + (-2A_2 - 6A_1)x - 6A_2 x^2 = 1 - 3x^2$$

$$\Rightarrow \left. \begin{aligned} 2A_2 - A_1 - 6A_0 &= 1 \\ -2A_2 - 6A_1 &= 0 \\ -6A_2 &= -3 \end{aligned} \right\} \Rightarrow \begin{aligned} A_2 &= \frac{-3}{-6} = \frac{1}{2} \\ A_1 &= -\frac{1}{6} \\ A_0 &= \frac{1}{36} \end{aligned}$$

$$\therefore y_p = \frac{1}{2}x^2 - \frac{1}{6}x + \frac{1}{36}$$

Note that y_h and y_p are Linearly Independent, therefore, the General Solution is given by

$$y(x) = y_h + y_p = C_1 e^{3x} + C_2 e^{-2x} + \frac{1}{2}x^2 - \frac{1}{6}x + \frac{1}{36}$$

Example: solve $y'' = 3t^2$.

$$\text{For } y_h: r^2 = 0 \Rightarrow r_{1,2} = 0$$

$$\therefore y_h(t) = C_1 + C_2 t$$

$$\text{Now, let } y_p(t) = (At^2 + Bt + C)t^2 = At^4 + Bt^3 + Ct^2$$

$$y_p' = 4At^3 + 3Bt^2 + 2Ct, \quad y_p'' = 12At^2 + 6Bt + 2C$$

$$\Rightarrow 12At^2 + 6Bt + 2C = 3t^2 \Rightarrow \begin{cases} C = 0, & B = 0 \\ A = \frac{1}{4} \end{cases}$$

$$\therefore y_p(t) = \frac{1}{4}t^4$$

$$\therefore \text{General Solution is } y(t) = C_1 + C_2 t + \frac{1}{4}t^4$$

ii) If $g(t) = C_1 e^{at}$, then we let $y_p = A e^{at}$.

We substitute y_p in the D.E, then find A

such that y_p is L.I with y_h .

Otherwise, we multiply by t^s , with $s=0,1,2$.

Example: solve $y'' - 3y' - 4y = 3e^{2t}$

Sol: y_h : $r^2 - 3r - 4 = 0 \Leftrightarrow (r+1)(r-4) = 0$

$\therefore r = -1$ & $r = 4 \Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{4t}$

Now, Let $y_p = A e^{2t} \Rightarrow y_p' = 2Ae^{2t} \Rightarrow y_p'' = 4Ae^{2t}$
 y_p & y_h are L.I

$$\Rightarrow 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) = 3e^{2t}$$

$$\Rightarrow 4A - 6A - 4A = 3 \Rightarrow -6A = 3 \Rightarrow \boxed{A = -\frac{1}{2}}$$

$$\therefore y_p(t) = -\frac{1}{2} e^{2t}$$

\therefore General solution is

$$y(t) = y_h + y_p = C_1 e^{-t} + C_2 e^{4t} - \frac{1}{2} e^{2t}.$$

Example: Solve $y'' - 3y' - 4y = 3e^{2t}$

with $y(0) = 0$ & $y'(0) = 2$

Sol: $y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2} e^{2t}$ (previous Example)

$$\Rightarrow y(0) = 0 \Rightarrow 0 = c_1 + c_2 - \frac{1}{2}$$

$$y'(0) = 2 : y'(t) = -c_1 e^{-t} + 4c_2 e^{4t} - e^{2t}$$

$$\therefore y'(0) = 2 \Rightarrow 2 = -c_1 + 4c_2 - 1$$

Solving the two equations: $c_1 = -\frac{1}{5}$, $c_2 = \frac{7}{10}$

$$\therefore y(t) = -\frac{1}{5} e^{-t} + \frac{7}{10} e^{4t} - \frac{1}{2} e^{2t}$$

Example: Solve $y'' - y' - 6y = 7e^{3x}$

Sol: $r^2 - r - 6 = 0 \Leftrightarrow (r+2)(r-3) = 0 \Leftrightarrow r_1 = 3, r_2 = -2$

$$\therefore y_h(t) = c_1 e^{3x} + c_2 e^{-2x}$$

Now, Let $y_p = A e^{3x} \cdot x$

$$y_p' = A e^{3x} + 3Ax e^{3x}, \quad y_p'' = 9Ax e^{3x} + 6A e^{3x}$$

$$\Rightarrow 9Ax e^{3x} + 6A e^{3x} - (A e^{3x} + 3Ax e^{3x}) - 6(Ax e^{3x}) = 7e^{3x}$$

$$\Rightarrow y_p = \frac{7}{5} x e^{3x} \Rightarrow y(t) = c_1 e^{3x} + c_2 e^{-2x} + \frac{7}{5} x e^{3x}$$

iii) If $g(t) = A \cos bx + B \sin bx$.

Then we assume $y_p = C \cos bx + D \sin bx$, then

substitute y_p in the D.E making sure that

y_h and y_p are L.I. Otherwise, we multiply

by t^s with $s = 0, 1, 2$.

Example: Solve $y'' - 3y' - 4y = 2 \sin t$

Sol: $r^2 - 3r - 4 = 0 \Leftrightarrow r_1 = -1, r_2 = 4$

$$\therefore y_h(t) = c_1 e^{-t} + c_2 e^{4t}$$

\downarrow L.I

Let $y_p = A \sin t + B \cos t$

$$y_p' = A \cos t - B \sin t$$

$$y_p'' = -A \sin t - B \cos t$$

$$\Rightarrow (-A \sin t - B \cos t) - 3(A \cos t - B \sin t) - 4(A \sin t + B \cos t) = 2 \sin t$$

$$\Rightarrow (-A + 3B - 4A) \sin t + (-B - 3A - 4B) \cos t = 2 \sin t$$

$$\Rightarrow \left. \begin{array}{l} -5A + 3B = 2 \\ -5B - 3A = 0 \end{array} \right\} \Rightarrow \boxed{A = \frac{-5}{17}}, \boxed{B = \frac{3}{17}}$$

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$$\therefore y_p = -\frac{5}{17} \sin t + \frac{3}{17} \cos t$$

$$\therefore y(t) = \underline{C_1 e^{-t} + C_2 e^{4t} - \frac{5}{17} \sin t + \frac{3}{17} \cos t}$$

Example: $y'' + y = 5 \sin t$

Sol: $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$$\therefore y_h(t) = C_1 \cos t + C_2 \sin t$$

↑ L.I

Let $y_p = (A \sin t + B \cos t) \cdot t$

$$\Rightarrow y_p' = (A \sin t + B \cos t) + t(A \cos t - B \sin t)$$

$$y_p'' = A \cos t - B \sin t + t(-A \sin t - B \cos t) + (A \cos t - B \sin t)$$

Substitute in the D.E:

$$2A \cos t - 2B \sin t - A t \sin t - B t \cos t = 5 \sin t$$

$$\Rightarrow -2B = 5 \Rightarrow B = -\frac{5}{2}$$

$$\therefore y_p = -\frac{5}{2} t \cos t,$$

\therefore General solution is

$$y(t) = C_1 \cos t + C_2 \sin t - \frac{5}{2} t \cos t.$$

iv) If $g(t) = P_n(t) e^{\alpha t}$, then we assume

$$y_p = t^s (A_0 + A_1 t + \dots + A_n t^n) e^{\alpha t}, \quad s=0,1,2$$

v) If $g(t) = P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$, then we assume

$$y_p = t^s \left[(A_0 + A_1 t + \dots + A_n t^n) e^{\alpha t} \cos \beta t + (B_0 + B_1 t + \dots + B_n t^n) e^{\alpha t} \sin \beta t \right]$$

with $s=0,1,2$.

Example: Find the particular solution for the following:

① $y'' - 3y' - 4y = -8e^t \cos 2t$

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t}$$

$$y_p = \left[A e^t \cos 2t + B e^t \sin 2t \right] t^0$$

② $y'' - 3y' - 4y = 3e^{2t} + 2 \sin t$

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t}$$

For y_p we have two cases:

$$(i) \quad y'' - 3y' - 4y = 3e^{2t} \Rightarrow y_{p_1} = Ae^{2t}$$

$$(ii) \quad y'' - 3y' - 4y = 2\sin t \Rightarrow y_{p_2} = (B\sin t + C\cos t)t$$

$$\therefore y_p = y_{p_1} + y_{p_2} = Ae^{2t} + B\sin t + C\cos t.$$

$$(3) \quad y'' + y = t + t\sin t$$

$$y_h(t) = C_1 \cos t + C_2 \sin t$$

We have two sub differential equations

$$(i) \quad y'' + y = t \Rightarrow y_{p_1} = (At + B)t^0$$

$$(ii) \quad y'' + y = t\sin t$$

$$\Rightarrow y_{p_2} = [(ct + D)\sin t + (Et + F)\cos t]t$$

$$\therefore y_p = y_{p_1} + y_{p_2} = (At + B)t + t[(ct + D)\sin t + (Et + F)\cos t]$$

$$\underline{(At + B) + (ct^2 + Dt)\sin t + (Et^2 + Ft)\cos t}$$

$$(4) \quad y'' + 2y' + 2y = t\sin t + 3te^{-t} + 4e^{-t}\sin t$$

$$y_h(t) = C_1 e^{-t}\sin t + C_2 e^{-t}\cos t$$

Now, we have three sub differential equations:

$$(i) \quad y'' + 2y' + 2y = t\sin t \Rightarrow y_{p_1} = [(At + B)\sin t + (ct + D)\cos t]t$$

$$(ii) \quad y'' + 2y' + 2y = 3t e^{-t}$$

$$\Rightarrow y_{p_2} = \left[(Et + F) e^{-t} \right] t^0$$

$$(iii) \quad y'' + 2y' + 2y = 4 e^{-t} \sin t$$

$$\Rightarrow y_{p_3} = \left[G e^{-t} \sin t + H e^{-t} \cos t \right] t^1$$

$$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} = \dots$$

Q13) (page 184) Find General Solution for:

$$y'' + y' + 4y = 2 \sinh t.$$

$$\text{Sol: } r^2 + r + 4 = 0 \Rightarrow r_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$

$$\Rightarrow y_h(t) = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{15}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{15}}{2}t\right)$$

$$\text{Now: } 2 \sinh t = 2 \left(\frac{e^t - e^{-t}}{2} \right) = e^t - e^{-t}$$

\therefore We have two sub differentials:

$$(i) \quad y'' + y' + 4y = e^t \Rightarrow y_{p_1} = A e^t$$

$$(ii) \quad y'' + y' + 4y = e^{-t} \Rightarrow y_{p_2} = B e^{-t}$$

$$\therefore y_p = y_{p_1} + y_{p_2} = A e^t + B e^{-t}$$

$$y_p' = A e^t - B e^{-t}, \quad y_p'' = A e^t + B e^{-t}$$

$$\Rightarrow Ae^t + Be^{-t} + Ae^t - Be^{-t} + 4Ae^t + 4Be^{-t} = e^t - e^{-t}$$

$$\Rightarrow 6Ae^t + 4Be^{-t} = e^t - e^{-t}$$

$$\Rightarrow 6A = 1 \Rightarrow \boxed{A = \frac{1}{6}} \quad \& \quad 4B = -1 \Rightarrow \boxed{B = -\frac{1}{4}}$$

Therefore, $y(t) = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{15}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$.

Q25) (page 184) Determine a suitable form for $y_p(t)$

for $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$.

Sol: $y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$.

Now we have three sub-differentials:

(i) $y'' - 4y' + 4y = 2t^2 \Rightarrow y_{p_1} = At^2 + Bt + C$

(ii) $y'' - 4y' + 4y = 4te^{2t} \Rightarrow (Dt + E)e^{2t} \cdot \boxed{t^2} = y_{p_2}$

(iii) $y'' - 4y' + 4y = t \sin 2t \Rightarrow$

$\Rightarrow y_{p_3} = (Ft + E) \sin 2t + (Gt + H) \cos 2t$

3.6 Variation of parameters

Consider the 2nd order linear differential equation:

$$y'' + p(t)y' + q(t)y = g(t) \quad \dots (1)$$

We studied the case when p and q are constants and $g(t)$ is one of the functions: \exp , poly , \sin , \cos , or finite sums and products of these functions.

Question: How can we solve eq.(1) if $g(x)$ is any other function, or p or q is nonconstant?

Ans: We will use the method of variation of parameters.

Thm 3.6.1: Consider the D.E

$$y'' + p(t)y' + q(t)y = g(t)$$

If p , q and g are continuous on an open interval

I , and if the functions y_1 and y_2 are

fundamental set of solutions of the homogeneous

D.E : $y'' + p(t)y' + q(t)y = 0$, then

the general solution of Eq. (1) is

$$y(t) = y_h + y_p$$
$$= \underbrace{c_1 y_1 + c_2 y_2}_{y_h} + \underbrace{v_1 y_1 + v_2 y_2}_{y_p}, \text{ where}$$

$$v_1 = - \int \frac{y_2(t)g(t)}{w(y_1, y_2)(t)} dt, \quad v_2 = \int \frac{y_1(t)g(t)}{w(y_1, y_2)(t)} dt$$

Example: Find the general solution of the D.E

$$y'' + 4y = 3 \csc t.$$

Sol: $y_h(t) = c_1 \cos 2t + c_2 \sin 2t \quad (r_{1,2} = \pm 2i)$

Let $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = \boxed{2} \neq 0$$

$$y_p = v_1 y_1 + v_2 y_2$$
$$= v_1 \cos 2t + v_2 \sin 2t.$$

$$V_1 = - \int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)} dt = - \int \frac{\sin 2t (3 \csc t)}{2} dt$$

$$= - \int \frac{\cancel{2} \sin t \cos t}{\cancel{2}} \cdot \frac{3}{\cancel{\sin t}} dt = -3 \int \cos t dt$$

$$= -3 \sin t.$$

$$V_2 = \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} dt = \int \frac{(\cos 2t)(3 \csc t)}{2} dt$$

$$= \frac{3}{2} \int (1 - 2 \sin^2 t) \cdot \frac{1}{\sin t} dt = \frac{3}{2} \int (\csc t - 2 \sin t) dt$$

$$= \frac{3}{2} \ln |\csc t - \cot t| + 3 \cos t$$

$$\therefore y_{op} = (-3 \sin t)(\cos 2t) + \left[\frac{3}{2} \ln |\csc t - \cot t| + 3 \cos t \right] (\sin 2t)$$

⇒ The General Solution is

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \underbrace{-3 \sin t \cos 2t}_{(*)} + \frac{3}{2} \sin 2t \ln |\csc t - \cot t| + \underbrace{3 \cos t \sin 2t}_{(**)}.$$

Note: $3 (\cos t (2 \sin t \cos t) - \sin t (2 \cos^2 t - 1))$ ↓

$(*)$ & $(**)$

$$= 3 [2 \sin t \cos^2 t - 2 \sin t \cos^2 t + \sin t] = \boxed{3 \sin t} \quad (150)$$

Example: Solve $x^2 y'' - 3x y' + 4y = x^2 \ln x, x > 0$

Sol: $y_h(x) : x^2 y'' - 3x y' + 4y = 0$ (Euler Eq.)

Let $t = \ln x$ with $A=1, B=-3, C=4$.

Then $\frac{d^2 y}{dt^2} + (-3-1) \frac{dy}{dt} + 4y = 0$

$$\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = 0$$

Charac. Eq.: $r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Leftrightarrow r_{1,2} = 2$

$$\begin{aligned} \Rightarrow y_h(t) &= c_1 e^{2t} + c_2 t e^{2t} \\ &= c_1 e^{2 \ln x} + c_2 (\ln x) e^{2 \ln x} \\ &= c_1 x^2 + c_2 (\ln x) x^2. \end{aligned}$$

$\Rightarrow y_1 = x^2$ and $y_2 = x^2 \ln x$

Standard form: $y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x, x > 0$

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x \ln x \end{vmatrix} = x^3 + 2x^3 \ln x - 2x^3 \ln x \\ &= x^3 \neq 0, \text{ since } x > 0. \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2 = v_1 x^2 + v_2 x^2 \ln x$$

$$v_1 = - \int \frac{y_2(x) g(x)}{W(y_1, y_2)(x)} dx = - \int \frac{(x^2 \ln x) \ln x}{x^3} dx$$

$$= - \int \frac{(\ln x)^2}{x} dx = - \int u^2 du$$

Let $u = \ln x$
 $du = \frac{1}{x} dx$

$$= - \frac{u^3}{3} = - \frac{(\ln x)^3}{3}$$

$$v_2 = \int \frac{y_1(x) g(x)}{W(y_1, y_2)} dx = \int \frac{(x^2) \ln x}{x^3} dx = \int \frac{\ln x}{x} dx$$

$$= \int u du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$\therefore y_p = - \frac{(\ln x)^3}{3} x^2 + \frac{(\ln x)^2}{2} x^2 \ln x$$

$$= \frac{1}{6} x^2 (\ln x)^3$$

$$\therefore y(t) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3$$

"End of Chapter 3"