Chapter 4. Higher Order Linear Equations.

4.1 General Theory of the nth order Linear Equations

Def: An nth order linear differential equation is an equation of the form.

$$L[y] := y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0 y = g(t). \quad \cdots (1)$$

with Corresponding homogeneous D.E.

$$L[y] = y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0 y = 0 \quad \cdots (2)$$

Eq. (1) needs nth Initial Conditions

$$y(t_0) = y_0, \; y'(t_0) = y_0', \; \cdots, \; y^{(n-1)}(t_0) = y_0^{(n-1)} \quad \cdots (3)$$

Theorem (4.1.1). If the functions $a_{n-1}(t), \cdots, a_1(t), a_0(t)$ and $g(t)$ are all continuous on the open Interval $I = (\alpha, \beta)$ such that $t_0 \in I$, then there exists exactly one Solution $y = \phi(t)$ of the D.E (1) that also satisfies the Initial Conditions (3).

Example: Determine the largest interval in which the solution of the following IVP is certain to exist.

$$(x-1)\,y^{(4)} + (x+1)\,y'' + (\tan x)\,y = 0$$

with $y(0)=1$, $y'(0) = y''(0) = y'''(0) = 0$

Sol: $y^{(4)} + \dfrac{(x+1)}{(x-1)}\,y'' + \dfrac{\tan x}{(x-1)}\,y = 0$

$\Rightarrow$ Interval : $\mathbb{R} \setminus \{1, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$

$\Rightarrow$ The largest Interval containing $t_0 = 1$ is $\left(-\frac{\pi}{2}, 1\right)$

The General solution for the homogeneous eq. (2) is given by: $y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

where $y_1, \dots, y_n$ are solutions of Eq. (2), and $c_1, c_2, \dots, c_n$ are arbitrary constants.

To find $c_1, \dots, c_n$, we use the Initial conditions in eq. (3).

Theorem (4.1.2): If the functions $a_0(t), \cdots, a_{n-1}(t)$ in eq. (2) are continuous on an open interval $I = (\alpha, \beta)$, and if the functions $y_1, \cdots, y_n$ are solutions of Eq (2) with $W(y_1, \cdots, y_n)(t_0) \neq 0$ for some $t_0 \in I$, then every solution of Eq (2) can be expressed as a linear combination of $y_1, \cdots, y_n$.

Def: A set of solutions $y_1, y_2, \cdots, y_n$ of eq. (2) whose Wronskian is nonzero is referred to as a Fundamental set of solutions (FFS).

Example: show that $\{1, t, t^3\}$ form a FSS of

$$t y^{(3)} - y'' = 0.$$

Sol: Let $y_1 = 1, \quad y_2 = t, \quad y_3 = t^3$

$\Rightarrow$ For $y_1 = 1 \quad , \quad y_1' = 0 \quad , \quad y_1'' = 0 \quad , \quad y_1'''(0) = 0$

$\Rightarrow \quad t(0) - 0 = 0 \quad (\checkmark) \quad \Rightarrow y_1$ is a solution.

Now, $y_2 = t$, $y_2' = 1$, $y_2'' = 0$, $y_2''' = 0$

$\Rightarrow$ $t(0) - 0 = 0$ ($\checkmark$) $\Rightarrow$ y_2 is a solution

$y_3 = t^3$, $y_3' = 3t^2$, $y_3'' = 6t$, $y_3''' = 6$

$\Rightarrow$ $t(6) - 6t = 0$ ($\checkmark$) $\Rightarrow$ y_3 is a solution

$$W(1, t, t^3) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t$$

$W(1, t, t^3)(1) = 6(1) = 6 \neq 0$

$\Rightarrow$ $\{1, t, t^3\}$ form a (FSS).

Linear Dependence and Independent.

Def: A functions $f_1, f_2, \dots, f_n$ are linearly independent if

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0.$$

Def: A function $f_1, \dots, f_n$ are Linearly Dependent if there exists $c_1, \dots, c_n$ not all zero such that

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

Example: $f_1 = 1$, $f_2 = 2+t$, $f_3 = 3-t^2$ are linearly independent, since if

$$c_1(1) + c_2(2+t) + c_3(3-t^2) = 0$$

$$(c_1 + 2c_2 + 3c_3) + c_2 t - c_3 t^2 = 0$$

$$\Rightarrow c_1 + 2c_2 + 3c_3 = 0 \quad \& \quad c_2 = 0, \quad c_3 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow \text{L. Independent.}$$

Example: Determine whether the functions

$$f_1(t) = 1, \quad f_2(t) = t+2, \quad f_3(t) = 3-t^2, \quad f_4(t) = t^2 + 4t$$

are Linearly Independent or dependent.

Sol: Let $c_1(1) + c_2(t+2) + c_3(3-t^2) + c_4(t^2+4t) = 0$

$$\Rightarrow (c_1 + 2c_2 + 3c_3) + (c_2 + 4c_4)t + (c_4 - c_3)t^2 = 0$$

$$\Rightarrow \quad c_1 + 2c_2 + 3c_3 = 0$$

$$c_2 + 4c_4 = 0$$

$$-c_3 + c_4 = 0$$

These three equations with four unknowns, have many solutions.

Since, if we set $c_4 = t \Rightarrow c_3 = t \Rightarrow c_2 = -4t$

$$\Rightarrow c_1 = 5t, \quad t \in \mathbb{R}$$

Thus $\{f_1, f_2, f_3, f_4\}$ are Linearly dependent.

4.2 Homogeneous equations with Constant Coefficients.

Consider the nth order linear homogeneous D.E:

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad \cdots (1)$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

To solve eq.(1), we use the same our knowledge of 2nd order linear D.Es considered in Ch3.

Example: Solve $2y''' - 4y'' - 2y' + 4y = 0$

Characteristic eq.: $2r^3 - 4r^2 - 2r + 4 = 0$

$$\Rightarrow \quad 2r^2(r-2) - 2(r-2) = (r-2)(2r^2-2) = 0$$

$$\Rightarrow \quad r = 2, 1, -1$$

$$\therefore \quad y_h = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}.$$

Example: Solve $y^{(4)} + 2y'' + y = 0$

$\Rightarrow r^4 + 2r + 1 = 0 \iff (r^2+1)(r^2+1) = 0$

$\therefore r_{1,2} = \pm i$ and $r_{3,4} = \pm i$

$\therefore y(t) = \left(c_1 e^{0t}\cos t + c_2 e^{0t}\sin t\right) + \left(c_3 e^{0}\cos t + c_4 \sin t\right)\underline{t}$

$\Rightarrow y(t) = c_1 \cos t + c_2 \sin t + C_3 t \cos t + C_4 t \sin t.$

Example: Solve $y^{(4)} - y = 0 \Rightarrow r^4 - 1 = 0$

$\iff (r^2+1)(r^2-1) = 0 \iff r_{1,2} = \pm 1,\ r_{3,4} = \pm i$

$y(t) = c_1 e^{t} + c_2 e^{-t} + C_3 \cos t + C_4 \sin t.$

Example: Solve $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

Sol: Char. Eq. : $r^4 + r^3 - 7r^2 - r + 6 = 0$

$= (r-1)(r^3 + 2r^2 - 5r - 6)$

$= (r-1)(r+1)(r^2 + r - 6)$

$= (r-1)(r+1)(r-2)(r+3) = 0 \Rightarrow$

$r = 1, -1, 2, -3$

$\therefore y(t) = c_1 e^{t} + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-3t}.$

$$
\begin{array}{r|l}
 & r^3 + 2r^2 - 5r - 6 \\
\hline
(r-1) & r^4 + r^3 - 7r^2 - r + 6 \\
 & -r^4 \pm r^3 \\
\hline
 & 2r^3 - 7r^2 - r + 6 \\
 & -2r^3 \pm 2r^2 \\
\hline
 & -5r^2 - r + 6 \\
 & +5r^2 \mp 5r \\
\hline
 & -6r + 6 \\
 & -6r + 6 \\
\hline
 & 0
\end{array}
$$

4.3. NonHomogeneous equations with Constant Coefficients.

(Method of undetermined Coefficients).

Consider the nth order linear nonhomogeneous eq. with Constant coefficients:

$$L[y] = a_n y^{(n)} + \dots + a_1 y' + a_0 y = g(t) \quad \dots (1)$$

We will use the method of undetermined Coefficients to find y_p if g is Constant, sin, cos, exp, polynomial or finite sums and products of these functions. as we did in Section 3.5.

Example: Solve $y''' - 3y'' + 3y' - y = 4e^t$

Charc. eq: $r^3 - 3r^2 + 3r - 1 = 0$

$\Rightarrow (r^3 - 1) - 3r(r-1) = 0$

$\Rightarrow (r-1)(r^2+r+1) - 3r(r-1) = 0$

$\Rightarrow (r-1)(r^2+r+1-3r) = 0$

$\Rightarrow (r-1)(r^2-2r+1) = 0$

$\Rightarrow (r-1)(r-1)(r-1) = 0.$

$\therefore\ y_h = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$

Let $y_p = A e^t \cdot \boxed{t^3} = A t^3 e^t$

To find y_p, find y_p''', y_p'', y_p' and substitute in the D.E.

Example: $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

y_h : $(r^4 + 2r + 1) = 0 \iff (r^2+1)^2 = 0$

$\therefore\ r_{1,2} = \pm i \quad , r_{3,4} = \pm i$

$\therefore\ y_h = c_1 \cos t + c_2 \sin t + c_3 t\cos t + c_4 t \sin t.$

Let $y_p = (A\sin t + B\cos t)\cdot \boxed{t^2}$

$\Rightarrow y_p = A t^2 \sin t + B t^2 \cos t.$

Example: $y''' - 4y' = t + 3\cos t + e^{-2t}$.

y_h : $r^3 - 4r = r(r^2-4) = r(r-2)(r+2) = 0$

$\therefore r = 0, 2, -2 \Longrightarrow y_h = c_1 + c_2 e^{2t} + c_3 e^{-2t}$.

To find y_p we have 3 subdifferential equations

1) $y''' - 4y' = t \Longrightarrow y_{p_1} = (At + B).\boxed{t}$

2) $y''' - 4y' = 3\cos t \Longrightarrow y_{p_2} = (C\sin t + D\cos t).\boxed{t^0}$

3) $y''' - 4y' = e^{-2t} \Longrightarrow y_{p_3} = E e^{-2t}.\boxed{t}$

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} = At^2 + Bt + C\sin t + D\cos t + Ete^{-2t}$

Example. $y^{(5)} + 4y''' = \cos^2 t - \sin^2 t = \cos(2t)$

y_h : $r^5 + 4r^3 = r^3(r^2+4) = 0 \Rightarrow r = 0, 0, 0, \pm 2i$

$\Rightarrow y_h = c_1 + c_2 t + c_3 t^2 + c_4 \cos 2t + c_5 \sin 2t$.

Let $y_p = \left(A\cos(2t) + B\sin 2t\right).\boxed{t}$

Example: $y^{(4)} + y = t$

y_h : $r^4 + 1 = 0 \Rightarrow r^4 + 2r^2 + 1 = 2r^2$

$\Rightarrow (r^2+1)^2 = 2r^2$

$\Rightarrow (r^2+1)^2 - (\sqrt{2}\,r)^2 = 0$ فرق بين مربعين

$\therefore (r^2 + 1 - \sqrt{2}\,r)(r^2 + 1 + \sqrt{2}\,r) = 0.$

$\therefore \underbrace{(r^2 - \sqrt{2}\,r + 1)}_{\Downarrow} \underbrace{(r^2 + \sqrt{2}\,r + 1)}_{\Downarrow} = 0$

$r_{1,2} = \dfrac{\sqrt{2} \pm \sqrt{2-4}}{2}$, $r_{3,4} = \dfrac{-\sqrt{2} \pm \sqrt{2-4}}{2}$

$r_{1,2} = \dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}} i$, $r_{3,4} = \dfrac{-1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}} i$

$$\therefore y_h = c_1 e^{\frac{1}{\sqrt{2}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 e^{\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right) + c_3 e^{-\frac{1}{\sqrt{2}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + c_4 e^{-\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right)$$

Let $y_p = At + B$.

$y_p' = A$, $y_p'' = 0 = y_p''' = y_p'''' \ldots$

Example: $y^{(4)} + y''' = 1 - t^2 e^{-t}$

$y_h: \; r^4 + r^3 = r^3(r+1) = 0 \Rightarrow r = 0, 0, 0, -1$

$\therefore y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$

We have 2. subdifferential equations.

1) $y^{(4)} + y''' = 1 \Rightarrow y_{p_1} = A \cdot \boxed{t^3}$

2) $y^{(4)} + y''' = -t^2 e^{-t} \Rightarrow y_{p_2} = (Bt^2 + Ct + D)e^{-t} \cdot \boxed{t}$

$\therefore y_p = y_{p_1} + y_{p_2} = At^3 + (Bt^3 + Ct^2 + Dt)\, e^{-t}.$

Example: $y^{(4)} + 2y^{(3)} + 2y'' = t\sin t + 3te^{-t} + 4e^{-t}\sin t.$

$y_h = c_1 + c_2 t + c_3 e^{-t}\cos t + c_4 e^{-t}\sin t$ (check!)

We have 3 subdifferential equations

1. $y^{(4)} + 2y^{(3)} + 2y'' = t\sin t \Rightarrow y_{p_1} = (At+B)\cos t + (Ct+D)\sin t$
2. $y^{(4)} + 2y^{(3)} + 2y'' = 3te^{-t} \Rightarrow y_{p_2} = (Et+F)e^{-t}$
3. $y^{(4)} + 2y^{(3)} + 2y'' = 4e^{-t}\sin t \Rightarrow y_{p_3} = (Ge^{-t}\sin t + He^{-t}\cos t) \cdot \boxed{t}$

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} \; \ldots$

End of Chapter 4.