Chapter 5. Series Solutions of Second Order Linear Eqs

5.1 Review of power Series

In this Chapter, we discuss the use of power series to construct a fundemental sets of solutions y_1 and y_2 of 2nd order linear D.Es whose coefficients are functions of the Independent variable and we we write the solutions y_1 and y_2 in term of power series.

Recall: Some results about power series that we need.

(1) A power series about the point x_0 "center" has the form: $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. And it is said to converge at x if

$$\lim_{k \to \infty} \sum_{n=0}^{k} a_n (x - x_0)^n \text{ exists for that } x.$$

(2) The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to Converge

absolutely at a point x, if the series

$$\sum_{n=0}^{\infty} |a_n (x-x_0)^n| \quad \text{converges.}$$

(3) To test the absolute convergence for the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, we use the ratio test.

Ratio test: $$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right|$$

$$= |x-x_0| \underbrace{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}_{=L}$$

$$= |x-x_0| \cdot L,$$

then the power series converges absolutely if

① $|x-x_0| \cdot L < 1$.

② And diverges if $|x-x_0| \cdot L > 1$

③ And if $|x-x_0| \cdot L = 1$, then the test is inconclusive.

Example : For which values of x does the power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$ converge?

Sol: $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} (n+1)(x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right|$

$= |x-2| \lim_{n\to\infty} \frac{n+1}{n}$

converge

$= |x-1| \cdot 1 < 1$

$\Rightarrow \quad -1 < x-1 < 1 \quad \Leftrightarrow \quad 1 < x < 3$

At $x = 1$, then: $\sum_{n=1}^{\infty} (-1)^{n+1} n (-1)^n = \sum_{n=1}^{\infty} -n$

nth term test

which is diverge.

At $x = 3$, then: $\sum_{n=1}^{\infty} (-1)^{n+1} n (1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} n$

nth term test

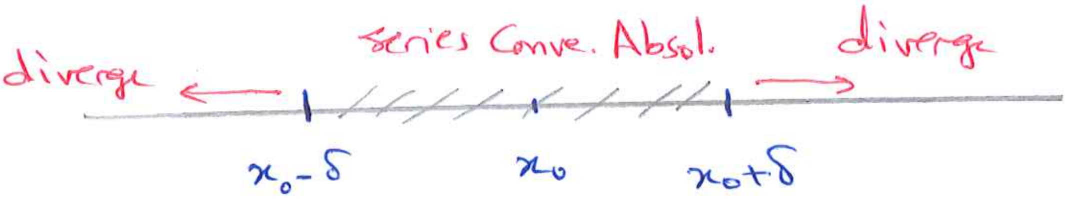
which is diverge, since $\lim_{n\to\infty} (-1)^{n+1} n \neq 0$

Alternating

Then the interval of absolute convergence is $(1, 3)$.

4) The radius of Convergence : is a positive number ρ such that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely for $|x - x_0| < \rho$, and diverges for $|x - x_0| > \rho$.

The Interval $|x - x_0| < \rho$ is called the Interval of Convergence



At $x_0 - \delta$ & $x_0 + \delta$, the series may Converges or diverges

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Example: Determine the radius of Convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$

previous example

sol: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} |x-2| \left(\frac{n+1}{n} \right) = |x-2| < 1$

center ρ

$\Rightarrow$ The radius of Convergence $= \rho = 1$.

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5) Differentiation and Integration of power series

If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

And $\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n (x - x_0)^{n+1}}{n+1} + C$

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6) The taylor series for the function f about $x = x_0$

is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$.

Note: A function f that has a Taylor series expansion about $x = x_0$ with radius of convergence $\rho > 0$ is said to be analytic at $x = x_0$,

Like: $\sin x$, $\cos x$, e^x, ...

7) Shifting of Index of Summation.

Example: Write the series $\sum_{n=2}^{\infty} (n+2)(n+1)\, a_n (x-1)^{n-2}$

as a series involves $(x-1)^n$.

Sol: Let $m = n-2 \Rightarrow n = m+2$

when $n = 2$, then $m = 0$

$$\therefore \sum_{m=0}^{\infty} (m+4)(m+3)\, a_{m+2} (x-1)^m .$$

(change m by n)

Example: Write the given expressions as a single sum involves x^n (Generic term involves x^n)

(1) $\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)\, a_{n+1} + 2a_n \right] x^n .$$

(2) $x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n) a_{n+1} + a_n \right] x^n .$$

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8) If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$

for all x in some open Interval with center x_0

then $a_n = b_n \ , \ \forall n = 0, 1, 2, \ldots$

In particular, if $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0 \ , \ \forall x$

then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

page 253. Rewrite the given expression as a sum whose generic term includes x^n.

(24) $$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

sol: $$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right]x^n$$

(27) $$x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

sol: $$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n + n(n+1)a_{n+1}\right]x^n$$

5.2 Series Solutions Near an ordinary points.

Part I :

In ch3, we described methods of solving second order linear D.Es with Constant Coefficients.

We now Consider methods of solving second order linear D.Es when the Coefficients are functions of the independent variable t. (or x)

It is sufficient to Consider the homogeneous eq.

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \qquad \cdots (1)$$

since the procedure for the nonhomogeneous Eq. is similar.

Def: A point x_0 such that $P(x_0) \neq 0$ in Eq. (1) is called an ordinary point.

If $P(x_0) = 0$, then x_0 is called Singular Point.

We will assume that P, Q and R in Eq. (1) are continuous. It follows that there is an interval about x_0 in which $P(x)$ is never zero.

In that interval, Eq. (1) can be written as

$$y'' + p(x)y' + q(x)y = 0 \qquad \ldots (2)$$

where $p(x) = \dfrac{Q(x)}{P(x)}$ and $q(x) = \dfrac{R(x)}{P(x)}$

are continuous functions. Therefore by Thm (3.2.1) there exists a unique solution that satisfies Eq. (1) together with the initial conditions

$$y(x_0) = y_0 \quad , y'(x_0) = y_0' .$$

To find such a solution in terms of power series, we assume that the solution has the form:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

which is defined on an interval of convergence

$|x - x_0| < \rho$ $\rightarrow$ radius of convergence, where x_0 = Ordinary point.

Example: Find Ordinary and Singular points of

1) $(x^2-x)y'' + xy' - 2x^2 y = 0$.

Sol: $P(x) = x^2 - x = x(x-1) = 0 \Rightarrow x = 0, x = 1$

are Singular points. So all other points real or Complex are Ordinary points.

2) $(x^2+4)y'' + xy = 0$

Sol: $P(x) = x^2+4 = 0 \Rightarrow x = \mp 2i$ are Singular points. All other points real or Complex are Ordinary.

Example: Find a series Solution of

$$y'' + y = 0 \quad , \quad -\infty < x < \infty .$$

Sol: $P(x) = 1 \neq 0$, then every point is an ordinary point, so we choose $x_0 = 0$ as a Simplest choice.

So, Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute y, y' & y'' in the D.E, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + a_n \right] x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0 \quad , \forall n = 0, 1, 2, \ldots$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \quad , \forall n = 0, 1, 2, \ldots$$

So, if $n = 0 \Rightarrow a_2 = \dfrac{-a_0}{(2)(1)} = \dfrac{-1}{2!} a_0$

if $n = 1 \Rightarrow a_3 = \dfrac{-a_1}{(3)(2)} = \dfrac{-a_1}{3!}$

if $n = 2 \Rightarrow a_4 = \dfrac{-a_2}{(4)(3)} = \dfrac{-(-a_0)}{(4)(3)(2)(1)}$

$$= \frac{+a_0}{4!}.$$

if $n=3 \Rightarrow a_5 = \dfrac{-a_3}{(5)(4)} = \dfrac{-(-a_1)}{(5)(4)(3!)} = \dfrac{a_1}{5!}$

$$\therefore\ y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \left(\frac{-a_0}{2!}\right)x^2 + \left(\frac{-a_1}{3!}\right)x^3 + \left(\frac{a_0}{4!}\right)x^4 + \left(\frac{a_1}{5!}\right)x^5 + \cdots$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{x^4}{4!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

Example: Find two linearly independent power series solutions y_1 and y_2 of $y'' - xy = 0$, $-\infty < x < \infty$, about the ordinary point $x_0 = 0$.

Sol: Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow (2)(1)a_2 x^0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

recurrence relation

$$\Rightarrow 2a_2 = 0 \quad \text{and} \quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0$$

for $n = 1, 2, \ldots$

So we have $a_2 = 0$, $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$, $n = 1, 2, \ldots$

If $n = 1 \Rightarrow a_3 = \dfrac{a_0}{(3)(2)} = \dfrac{a_0}{6}$.

If $n = 2 \Rightarrow a_4 = \dfrac{a_1}{(4)(3)} = \dfrac{a_1}{12}$.

If $n = 3 \Rightarrow a_5 = \dfrac{a_2}{(5)(4)} = 0$.

If $n = 4 \Rightarrow a_6 = \dfrac{a_3}{(6)(5)} = \dfrac{a_0}{(6)(5)(6)} = \dfrac{a_0}{180}$.

If $n = 5 \Rightarrow a_7 = \dfrac{a_4}{(7)(6)} = \dfrac{a_1}{(7)(6)(4)(3)} = \dfrac{a_1}{504}$.

$$\therefore\ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + 0 + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + 0 + \frac{a_0}{180} x^6 + \frac{a_1}{504} x^7 + \cdots$$

$$= a_0 \underbrace{\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right)}_{\Downarrow} + a_1 \underbrace{\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right)}_{\Downarrow}$$

$$= a_0 \quad y_1 \quad + a_1 \quad y_2$$

Now, to check that y_1 and y_2 are Linearly Independent

We compute $W(y_1, y_2)(0)$

$$\Rightarrow W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\therefore$ $\{y_1, y_2\}$ forms a Fundamental Set of Solution.

Example: Solve $y'' - xy = 0 \quad , \quad x_0 = 1$

Sol: Let $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n (x-1)^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-1)^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$(x-1)+1$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n (x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\Rightarrow (2)(1)\, a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1} - a_n \right] (x-1)^n = 0$$

$$\Rightarrow \quad a_2 = \frac{a_0}{2} \quad \text{and} \quad a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)} \; , \; n = 1, 2, \ldots$$

If $n = 1$, $\quad a_3 = \dfrac{a_1 + a_0}{(3)(2)} = \dfrac{a_0}{6} + \dfrac{a_1}{6}$

If $n = 2$, $\quad a_4 = \dfrac{a_2 + a_1}{(4)(3)} = \dfrac{a_1}{12} + \dfrac{a_0}{(4)(3)(2)}$

If $n = 3$, $\quad a_5 = \dfrac{a_3 + a_2}{(5)(4)} = \dfrac{a_3}{(5)(4)} + \dfrac{a_2}{(5)(4)}$

$$\Rightarrow \quad a_5 = \frac{a_0}{(6)(5)(4)} + \frac{a_1}{(6)(5)(4)} + \frac{a_0}{(5)(4)(2)} .$$

$$\vdots$$

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x-1)^3$$

$$+ \left(\frac{a_1}{12} + \frac{a_0}{24}\right)(x-1)^4 + \cdots$$

$$= a_0 \overbrace{\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right)}^{y_1}$$

$$+ a_1 \underbrace{\left((x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right)}_{y_2}$$

$$= a_0 y_1 + a_1 y_2$$

$$W(y_1, y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ y_1 and y_2 are Linearly Independent.

so $\{y_1, y_2\}$ forms a FSS.

Q9) page 263. Find a power series solution for

$$(1+x^2)y'' - 4xy' + 6y = 0, \quad \text{about } x_0 = 0.$$

Sol: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow (1+x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n} - \sum_{n=1}^{\infty} 4n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\Rightarrow (2)(1)a_2 + (3)(2)a_3 x - 4a_1 x + 6a_0 + 6a_1 x$$

$$+ \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} + \underbrace{n(n-1)a_n - 4na_n + 6a_n}_{(n^2-5n+6)}\Big]x^n = 0$$

$$\Rightarrow \quad 2a_2 + 6a_0 + (6a_3 + 2a_1)x +$$

$$+ \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n\Big]x^n = 0$$

$$\Rightarrow \quad 2a_2 + 6a_0 = 0 \quad , \quad 6a_3 + 2a_1 = 0$$

and $(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n = 0$, $n=2,\dots$

recurrence relation.

$$\Rightarrow \quad a_2 = -3a_0 \quad , \quad a_3 = -\frac{1}{3}a_1$$

$$\& \quad a_{n+2} = \frac{-(n-2)(n-3)}{(n+2)(n+1)}a_n \quad , \quad n = 2, 3, \dots$$

If $n=2$, $a_4 = 0$. If $n=3$, $a_5 = 0$

If $n=4$, $a_6 = -\frac{1}{15}(0) = 0$. If $n=5$, $a_7 = 0$ ...

$$\therefore \ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3 + 0 + 0 + \dots^{0}$$

$$= a_0\underbrace{(1 - 3x^2)}_{y_1} + a_1\underbrace{\left(x - \frac{x^3}{3}\right)}_{y_2}.$$

$$= a_0 \ y_1 + a_1 \ y_2$$

$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$