Chapter 6. The Laplace Transform.

6.1 Definition of the Laplace Transform.

Review : (Section 8.7 Calculus I)

Improper Integral is defined as:

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt \quad , \text{where } A > 0 \text{ (real)}$$

If $\int_a^A f(t)\,dt$ exists for $A > a$, and the limit

as $A \to \infty$ exists , then the improper integral is

said to converge to the limit value.

Otherwise the integral is said to be divrge.

Example: $\int_0^\infty e^{\alpha t}\,dt = \lim_{A\to\infty} \int_0^A e^{\alpha t}\,dt$

$$= \lim_{A\to\infty} \left.\frac{e^{\alpha t}}{\alpha}\right|_0^A = \lim_{A\to\infty} \frac{e^{\alpha A}}{\alpha} - \frac{1}{\alpha} \quad , \qquad \alpha \neq 0$$

$$= \lim_{A \to \infty} \frac{e^{\alpha A} - 1}{\alpha} = \begin{cases} -\frac{1}{\alpha} & , \alpha < 0 \\ \text{div.} & , \alpha > 0 \end{cases}$$

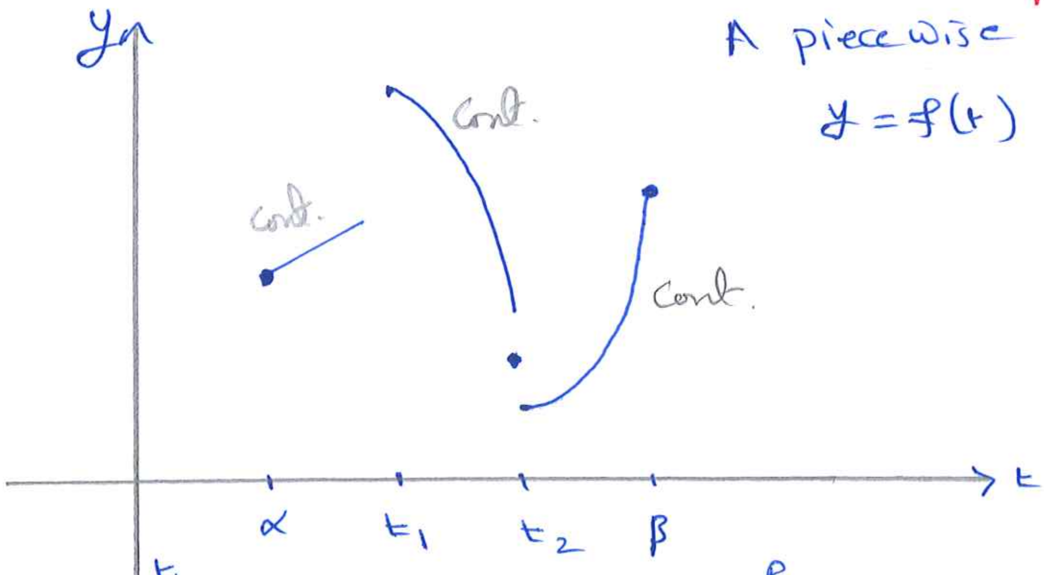
If $\alpha = 0 \Rightarrow \int_0^\infty e^{\alpha t}\, dt = \int_0^\infty 1\, dt = \infty$. (div.)

Example: $\int_1^\infty \frac{1}{t}\, dt = \lim_{A \to \infty} \int_1^A \frac{1}{t}\, dt$

$$= \lim_{A \to \infty} \ln |t| \Big|_1^A = \lim (\ln A - \ln 1) = \infty, \text{(div.)}$$

Def: A function f is said to be piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous except for a finite number of jump discontinuities.

Example:



A piecewise cont. function $y = f(t)$

$$\int_\alpha^\beta f(t)\, dt = \int_\alpha^{t_1} f(t)\, dt + \int_{t_1}^{t_2} f(t)\, dt + \int_{t_2}^{\beta} f(t)\, dt$$

Remark: If f is a piecewise continuous on $a \le t \le b$, then $\int_a^b f(t)\,dt$ exists. However, piecewise continuity is not enough to ensure convergence of $\int_a^b f(t)\,dt$. In this case we use comparison test.

$|f(t)| \le g(t)$. If $\int_M^\infty g(t)\,dt$ converges then $\int_a^\infty f(t)\,dt$ con.

If $f(t) \ge g(t)$ & $\int_a^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)$ div.

The Laplace Transform.

One of the tools to solve Linear Differential equations is Laplace Transform

Def: An integral transform is a relation of the form

$$F(s) = \int_\alpha^\beta f(t)\,K(s,t)\,dt, \qquad \cdots (1)$$

where $K(s,t)$ is called the kernal of the transformation and α, β are also given.

It is possible that $\alpha = -\infty$, or $\beta = \alpha$, or both.

Eq. (1) transforms f into another function F, which is called the transform of f.

Def: (Laplace Transform) [$\mathcal{L}$.T].

The Laplace transform of f is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \quad \cdots (2)$$

provided that the improper integral converges.

Thm: Existence of Laplace Transform:

suppose that

1) f is piecewise continuous on $0 \le t \le A$, $\forall A > 0$.

2) $|f(t)| \le K e^{at}$, $t \geq M$, where $K, a, M \in \mathbb{R}$ and K and M are positive. Then the Laplace transform $\mathcal{L}(f(t)) = F(s)$ defined by eq. (2) exists for $s > a$.

Examples:

1) $\mathcal{L}(1) = \int_0^\infty 1\, e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{-st}\, dt$

$$= \lim_{A\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^A = \lim_{A\to\infty} \frac{-e^{-sA}+1}{s} \quad , \quad s>0.$$

$$\Rightarrow \quad \mathcal{L}(1) = \frac{1}{s} \quad , \quad s>0.$$

In General $\mathcal{L}(k) = \frac{k}{s}$, $s>0$ and k is Constant

Ex. $\mathcal{L}(2023) = \frac{2023}{s} \quad , \quad s>0.$

2) $\mathcal{L}(e^{kt}) = \int_0^\infty e^{kt} e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{(k-s)t}\, dt$

$$= \lim_{A\to\infty} \left. \frac{e^{(k-s)t}}{(k-s)} \right|_0^A = \lim_{A\to\infty} \left. \frac{e^{-(s-k)t}}{-(s-k)} \right|_0^A$$

$$= \lim_{A\to\infty} \frac{1-e^{-(s-k)A}}{(s-k)} = \frac{1}{s-k} \quad , \quad s>k.$$

$$\Rightarrow \boxed{\mathcal{L}(e^{kt}) = \frac{1}{s-k}} \quad , \quad s > k$$

ex. $\mathcal{L}(e^{2t}) = \dfrac{1}{s-2} \quad , \quad s > 2.$

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3) $$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st}\,dt = \lim_{A\to\infty} \int_0^A t^2 e^{-st}\,dt$$

$$= \lim_{A\to\infty} \left(-\frac{t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2e^{-st}}{s^3} \right) \Bigg|_0^A$$

$$\begin{array}{ccc} t^2 & & e^{-st} \\ & + & \\ 2t & & -\dfrac{e^{-st}}{s} \\ & - & \\ 2 & & \dfrac{e^{-st}}{s^2} \\ & + & \\ 0 & & \dfrac{e^{-st}}{-s^3} \end{array}$$

$$= \lim_{A\to\infty} \left[\frac{-\frac{A^2}{s} - \frac{2A}{s^2} - \frac{2}{s^3}}{e^{sA}} + \frac{2}{s^3} \right]$$

$$\overset{L.H}{=} 0 + \frac{2}{s^3} = \frac{2!}{s^3} \quad , \quad s > 0$$

In General: $\boxed{\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}} \quad , \quad s > 0}$

ex. $\mathcal{L}(t^4) = \dfrac{4!}{s^5} \quad , \quad s > 0$

$\mathcal{L}(t) = \dfrac{1}{s^2} \quad , \quad s > 0$

4) $\mathcal{L}(\sin at) = \int_0^\infty \sin at \, e^{-st} \, dt$

$$= \lim_{A\to\infty} \int_0^A \sin at \, e^{-st} \, dt \quad \ldots (*)$$

Integrating by parts: $u = e^{-st}$ $\quad dv = \sin at \, dt$

$du = -s e^{-st} dt$, $V = -\frac{\cos at}{a}$

$-\int$

$$(*) \Rightarrow \lim_{A\to\infty} \left[-\frac{\cos at}{a} e^{-st} \Big|_0^A - \int_0^A \underline{\frac{s}{a} e^{-st} \cos at \, dt} \right] , s>0$$

Again by parts.

⋮

Continue.

$$\Rightarrow F(s) = \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt$$

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

$$\Rightarrow F(s)\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a}$$

$$\Rightarrow F(s) = \frac{1}{a} \cdot \frac{a^2}{s^2+a^2} = \frac{a}{s^2+a^2}$$

$$\therefore \mathcal{L}(\sin at) = \frac{a}{s^2+a^2} , \quad s>0$$

5) $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $s>0$.

6) $\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$, $s>|a|$

7) $\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$, $s>|a|$

8) $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$, $s>a$

9) $\mathcal{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$, $s>a$

10) $\mathcal{L}(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}}$, $n=0,1,2,\ldots$

11) $\mathcal{L}(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n}\left(\mathcal{L}(f(t))\right) = F^{(n)}(s)(-1)^n$

12) $\mathcal{L}(\alpha f(t) \pm \beta g(t)) = \alpha \mathcal{L}(f(t)) + \beta \mathcal{L}(g(t))$.

Proof 12 : $\int_0^\infty (\alpha f(t) \pm \beta g(t))\, e^{-st}\, dt$

$$= \alpha \int_0^\infty f(t)\, e^{-st}\, dt \pm \beta \int_0^\infty g(t)\, e^{-st}\, dt = \alpha \mathcal{L}(f(t)) \pm \beta \mathcal{L}(g(t))$$

Proof (7): $\mathcal{L}(\sinh at) = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right)$

$$= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a}\right) - \frac{1}{2}\left(\frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a-(s-a)}{s^2-a^2}\right)$$

$$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right) = \frac{a}{s^2-a^2} \quad , \quad s > |a|$$

Proof (8): $\mathcal{L}(\cosh(at)) = \mathcal{L}\left(\frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2}\left[\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})\right] = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2-a^2}\right) = \frac{s}{s^2-a^2} \quad , \quad |a| < s$$

Example: Find $\mathcal{L}(4e^{-2t} - 3\sin 4t)$

$$= 4\mathcal{L}(e^{-2t}) - 3\mathcal{L}(\sin 4t)$$

$$= \frac{4}{s+2} - \frac{3 \cdot 4}{s^2+(4)^2} = \frac{4}{s+2} - \frac{12}{s^2+16}.$$

Example: $\mathcal{L}(\sin^2 t) = \mathcal{L}\left(\frac{1-\cos 2t}{2}\right)$

$$= \mathcal{L}\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\mathcal{L}(\cos 2t)$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2} \cdot \frac{s}{s^2+4} = \frac{1}{2s} - \frac{s}{2(s^2+4)}.$$

Example: $\mathcal{L}(t^2 e^t) = (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}(e^t)\right) = F''(s)$

$$F(s) = \mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\Rightarrow F'(s) = \frac{-1}{(s-1)^2} = -(s-1)^{-2}$$

$$\Rightarrow F''(s) = 2(s-1)^{-3}$$

$$\Rightarrow \mathcal{L}(t^2 e^t) = 2(s-1)^{-3} = \frac{2}{(s-1)^3}.$$

Example: $\mathcal{L}(\cosh(6t)) = \frac{s}{s^2-36}$, $s > 6$

Example: $\mathcal{L}(e^{2t}\sin 6t) = \frac{6}{(s-2)^2+36}$

$a = 2$, $b = 6$ in Rule #8

Example: $\mathcal{L}\left(\sin\left(2t+\frac{\pi}{3}\right)\right)$

$$= \mathcal{L}\left(\sin 2t \cos\frac{\pi}{3} + \cos 2t \sin\frac{\pi}{3}\right)$$

$$= \mathcal{L}\left(\frac{1}{2}\sin 2t + \frac{\sqrt{3}}{2}\cos 2t\right)$$

$$= \frac{1}{2}\mathcal{L}(\sin 2t) + \frac{\sqrt{3}}{2}\mathcal{L}(\cos 2t).$$

$$= \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{\sqrt{3}}{2}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{1+\frac{\sqrt{3}}{2}s}{s^2+4}.$$

proof 5 $\mathcal{L}(\cos at) = \mathcal{L}\left(\frac{e^{ati}+e^{-ati}}{2}\right)$

$$= \frac{1}{2}\mathcal{L}\left(e^{ait}+e^{-ait}\right) = \frac{1}{2}\left(\frac{1}{s-ai}+\frac{1}{s+ai}\right)$$

$$= \frac{1}{2}\left(\frac{s+ai+s-ai}{s^2-(ai)^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

Note that: $\left.\begin{array}{l}\cos\theta = e^{i\theta} - i\sin\theta \\ \cos\theta = e^{-i\theta} + i\sin\theta\end{array}\right\} \Rightarrow$ Euler formulas

$$\Rightarrow \quad 2\cos\theta = e^{i\theta} + e^{-i\theta} \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

6.2 Solutions of Initial Value Problem.

In this section we will show how the Laplace transform can be used to solve IVPs for linear DE with Constant Coefficients.

The inverse of Laplace Transform.

$$\mathcal{L}(f(t)) = F(s) \Rightarrow f(t) = \mathcal{L}^{-1}(F(s))$$

Example: Find the Inverse of Laplace Transform.

1) $\mathcal{L}^{-1}\left(\frac{2023}{s}\right) = 2023$

2) $\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2!}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) = \frac{1}{2}t^2.$

3) $\mathcal{L}^{-1}\left(\frac{2s-3}{s^2+4}\right) = 2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$

$$= 2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \frac{3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= 2\cos 2t - \frac{3}{2}\sin 2t.$$

4) $\mathcal{L}^{-1}\left(\frac{1}{(s-4)^4}\right) = \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{3!}{(s-4)^4}\right)$

$= \frac{1}{3!} e^{4t} \mathcal{L}^{-1}\left(\frac{3!}{s^4}\right)$

$= \frac{1}{6} e^{4t} \cdot t^3.$

5) $\mathcal{L}^{-1}\left(\frac{s}{(s-2)^2+9}\right) = \mathcal{L}^{-1}\left(\frac{(s-2)+2}{(s-2)^2+9}\right)$

$= \mathcal{L}^{-1}\left(\frac{(s-2)}{(s-2)^2+9}\right) + \mathcal{L}^{-1}\left(\frac{2}{(s-2)^2+9}\right)$

$= e^{2t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \frac{2}{3} e^{2t}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$

$= e^{2t}\cos 3t + \frac{2}{3} e^{2t}\sin 3t.$

6) $\mathcal{L}^{-1}\left(\frac{2s+2}{s^2+2s+6}\right) = \mathcal{L}^{-1}\left(\frac{2(s+1)}{\underbrace{(s+1)^2+5}}\right)$ إكمال مربع

$= 2\mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+5}\right) = 2e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+5}\right)$

$= 2e^{-t}\cos(\sqrt{5}\,t).$

7) $\mathcal{L}^{-1}\left(\frac{1}{s^2-2s-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+1)}\right)$

Using Partial Fraction:

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$\Rightarrow A(s+1) + B(s-3) = 1$

or let
$s=-1 \Rightarrow B = -\frac{1}{4}$
$s=3 \Rightarrow A = \frac{1}{4}$
(cover up method for Linear factors)

$\Rightarrow (A+B)s + (A-3B) = 1$

$\Rightarrow \left.\begin{aligned} A+B &= 0 \\ A-3B &= 1 \end{aligned}\right\}$ $\boxed{B = -\frac{1}{4}}$ & $\boxed{A = \frac{1}{4}}$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2-2s-3}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{-\frac{1}{4}}{s+1}\right)$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{4}e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4}e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}.$$

8) $\mathcal{L}^{-1}\left(\dfrac{s}{(s+2)(s^2+4)}\right)$

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad s = A(s^2+4) + (Bs+C)(s+2)$$

$$\Rightarrow \quad s = As^2 + 4A + Bs^2 + 2Bs + Cs + 2C$$

$$\Rightarrow \quad s = (A+B)s^2 + (2B+C)s + (4A+2C)$$

$$\Rightarrow \quad \left.\begin{aligned} A+B&=0 \\ 2B+C&=1 \\ 4A+2C&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\tfrac{1}{4} \\ B &= \tfrac{1}{4} \\ C &= \tfrac{1}{2} \end{aligned}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{-\frac{1}{4}}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{4}s+\frac{1}{2}}{s^2+4}\right)$$

$$= -\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{2\cdot 2}\mathcal{L}^{-1}\left(\frac{1\cdot 2}{s^2+4}\right)$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t.$$

Laplace of Derivatives.

1) $\mathcal{L}\left(y(t)\right) = Y(s)$

2) $\mathcal{L}\left(y'(t)\right) = sY(s) - y(0)$

3) $\mathcal{L}\left(y''(t)\right) = s^2 Y(s) - s\,y(0) - y'(0)$

4) $\mathcal{L}\left(y'''(t)\right) = s^3 Y(s) - s^2 y(0) - s\,y'(0) - y''(0)$

$\vdots$

In General:

$$\mathcal{L}\left(y^{(n)}(t)\right) = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y''(0) - \dots - y^{(n-1)}(0).$$

Proof (2): $\mathcal{L}\left(y'(t)\right) = \lim\limits_{A\to\infty} \int_0^A e^{-st} y'(t)\,dt$ $\quad \left(u = e^{-st},\ dv = y'dt \ \vdots \ \text{by parts}\right)$

$$= \lim_{A\to\infty}\left(e^{-st} y(t)\Big|_0^A + \int_0^A s e^{-st} y(t)\,dt \right)$$

$$= \lim_{A\to\infty}\left(e^{-sA} y(A) - y(0)\right) + s \lim_{A\to\infty} \int_0^A e^{-st} y(t)\,dt$$

$$= -y(0) + s\,\mathcal{L}\left(y(t)\right) = sY(s) - y(0)$$

Proof (3): $\mathcal{L}(y''(t))$

Let $g(t) = y'(t)$, then $g'(t) = y''(t)$

$$\Rightarrow \mathcal{L}(g'(t)) = s\,\mathcal{L}(g(t)) - g(0)$$

$$= s\,\mathcal{L}(y'(t)) - y'(0)$$

$$= s\Big(s\,Y(s) - y(0)\Big) - y'(0)$$

$$= s^2\,Y(s) - s\,y(0) - y'(0)$$

Now, we are ready to use Laplace Transform to Solve IVP.

Example: Use Laplace Transform to solve:

$$y'' - y' - 2y = 0 \quad , y(0) = 1 \, , y'(0) = 0$$

Sol: Take Laplace Transform for both sides.

$$\mathcal{L}(y'' - y' - 2y) = \mathcal{L}(0)$$

$$\Rightarrow \mathcal{L}(y'') - \mathcal{L}(y') - 2\,\mathcal{L}(y) = 0$$

$$\Rightarrow \left[s^2 Y(s) - s y(0) - y'(0)\right] - \left[s Y(s) - y(0)\right] - 2Y(s) = 0$$

$$\Rightarrow [s^2 - s - 2]Y(s) - s\underbrace{y(0)}_{=1} + \underbrace{y(0)}_{1=} - \underbrace{y'(0)}_{0=} = 0$$

$$\Rightarrow [s^2 - s - 2]Y(s) = s - 1$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)}$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s-1}{(s+1)(s-2)}\right)$$

$$\frac{s-1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$\Rightarrow s - 1 = (s-2)A + (s+1)B.$$

Let $s = -1 \Rightarrow A = \frac{2}{3}$

Let $s = 2 \Rightarrow B = \frac{1}{3}$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{\frac{2}{3}}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s-2}\right)$$

$$= \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

Example: Solve $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$

Sol: $\mathcal{L}(y'' + y) = \mathcal{L}(\sin 2t)$

$$\Rightarrow \left[s^2 Y(s) - s\,y(0) - y'(0)\right] + Y(s) = \frac{2}{s^2+4}$$

$$\Rightarrow (s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$\Rightarrow (s^2+1)Y(s) = \frac{2}{s^2+4} + (2s+1)\cdot\left(\frac{s^2+4}{s^2+4}\right)$$

$$\Rightarrow (s^2+1)Y(s) = \frac{2 + 2s^3 + 8s + s^2 + 4}{(s^2+4)}$$

$$\Rightarrow Y(s) = \frac{(2s^3 + s^2 + 8s + 6)}{(s^2+1)(s^2+4)}$$

Using Partial Fraction:

$$\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{(As+B)}{(s^2+1)} + \frac{(Cs+D)}{(s^2+4)}$$

$$\Rightarrow 2s^3 + s^2 + 8s + 6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

$$\left.\begin{aligned} A + C &= 2 \\ B + D &= 1 \\ 4A + C &= 8 \\ 4B + D &= 6 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ B &= 5/3 \\ C &= 0 \\ D &= -2/3 . \end{aligned}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{2s + 5/3}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{-2/3}{s^2+4}\right)$$

$$= 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t .$$

Example : Solve $y' = t \quad , y(0) = 1$

Sol: $\mathcal{L}(y') = \mathcal{L}(t)$

$$\Rightarrow sY(s) - y(0) = \frac{1}{s^2}$$

$$\Rightarrow sY(s) = \frac{1}{s^2} + 1 = \frac{1+s^2}{s^2}$$

$$\Rightarrow Y(s) = \frac{1}{s^3} + \frac{1}{s}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\right) = \frac{1}{2}t^2 + 1$$

Example: Solve $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = -1$

Using Laplace Transform.

Sol: $\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(4e^{-t})$

$$\Rightarrow \left[s^2 Y(s) - s y(0) - y'(0)\right] + 2\left[s Y(s) - y(0)\right] + Y(s) = \frac{4}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1) Y(s) = \frac{4}{s+1} + 2s - 1 + 4$$

$$\Rightarrow \underbrace{(s^2 + 2s + 1)}_{(s+1)(s+1)} Y(s) = \frac{2s^2 + 5s + 7}{(s+1)}$$

$$\Rightarrow Y(s) = \frac{2s^2 + 5s + 7}{(s+1)^3}$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{2s^2 + 5s + 7}{(s+1)^3}\right)$$

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\Rightarrow 2s^2 + 5s + 7 = A(s+1)^2 + B(s+1) + C$$

$$= A(s^2 + 2s + 1) + (Bs + B) + C$$

$$\Rightarrow 2s^2+5s+7 = As^2+(2A+B)s+(A+B+C)$$

$$\Rightarrow \boxed{A=2}$$

$$2A+B=5 \Rightarrow \boxed{B=1}$$

$$A+B+C=7 \Rightarrow \boxed{C=4}$$

$$2\mathcal{L}^{-1}\left(\frac{2!}{(s+1)^3}\right)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{2}{s+1}\right)+\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)+\mathcal{L}^{-1}\left(\frac{4}{(s+1)^3}\right)$$

$$= 2e^{-t} + e^{-t}.t + 2e^{-t}.t^2$$

$$= (2+t+2t^2)e^{-t}.$$

Example: Solve $y^{(4)}-y=0$, $y(0)=y''(0)=y'''(0)=0$, $y'(0)=1$

Using Laplace transform.

Sol: $\mathcal{L}(y^{(4)})-\mathcal{L}(y)=\mathcal{L}(0)$

$$\Rightarrow s^4Y(s)-s^3\cancel{y(0)}^{0}-s^2y'(0)-s\cancel{y''(0)}^{0}-\cancel{y'''(0)}^{0}-Y(s)=0$$

$$\Rightarrow (s^4-1)Y(s) = s^2.$$

$$\therefore Y(s) = \frac{s^2}{(s^4-1)} = \frac{s^2}{(s-1)(s+1)(s^2+1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(Y(s)\right) = \mathcal{L}^{-1}\left(\frac{s^2}{(s-1)(s+1)(s^2+1)}\right)$$

$$\frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore \quad s^2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1)$$

Let $s = 1 \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

$s = -1 \Rightarrow 1 = -4B \Rightarrow \boxed{B = -\frac{1}{4}}$

$s = 0 \Rightarrow 0 = A + (-B) - D \Rightarrow \boxed{D = \frac{1}{2}}$

$s = 2 \Rightarrow 4 = 15A + 5B + (2C + \frac{1}{2})(3)$

$\Rightarrow \boxed{C = 0}$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{-\frac{1}{4}}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s^2+1}\right)$$

$$y(t) = \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t.$$

$$= \frac{\sinh t + \sin t}{2}$$

6.3. Step functions.

Some of the most interesting elementary applications of Laplace Transform occur in the solution of Linear differential equations with discontinuous functions.

1st shifting

Theorem: (6.3.2) (First Translation Theorem)

If $\mathcal{L}(f(t)) = F(s)$ and $a \in \mathbb{R}$, then:

$$\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f(t))_{s \to s-a} = F(s-a).$$

or $\mathcal{L}^{-1}(F(s-a)) = e^{at}\mathcal{L}^{-1}(F(s))$

Example: ① $\mathcal{L}(e^{5t} \cdot t^3) = \mathcal{L}(t^3)_{s \to s-5} = \dfrac{3!}{(s-5)^4}$.

② $\mathcal{L}(e^{-2t} \cos 4t) = \mathcal{L}(\cos 4t)_{s \to s+2} = \dfrac{s}{s^2+16}\Big|_{s \to s+2}$

$$= \frac{(s+2)}{(s+2)^2+16}$$

③ $$\mathcal{L}^{-1}\left(\frac{2s+5}{(s-3)^2}\right) = \mathcal{L}^{-1}\left(\frac{2(s-3)+6+5}{(s-3)^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{2(s-3)}{(s-3)^2} + \frac{11}{(s-3)^2}\right) = \mathcal{L}^{-1}\left(\frac{2}{(s-3)} + \frac{11}{(s-3)^2}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + 11\,\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = 2e^{3t}\cdot 1 + 11e^{3t}\cdot t.$$

$$= (2+11t)e^{3t}.$$

OR : $$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\Rightarrow \quad 2s+5 = A(s-3) + B.$$

Let $s = 3 \Rightarrow B = 11$

$s = 0 \Rightarrow A = 2$

$$\therefore \mathcal{L}^{-1}\left(\frac{2s+5}{(s-3)^2}\right) = \mathcal{L}^{-1}\left(\frac{2}{(s-3)} + \frac{11}{(s-3)^2}\right)$$

$$= 2e^{3t}\cdot 1 + 11e^{3t}\cdot t$$

$$= (2+11t)e^{3t}.$$

(4) $\mathcal{L}^{-1}\left(\dfrac{\frac{1}{2}s+5/3}{s^2+4s+6}\right) = \mathcal{L}^{-1}\left(\dfrac{\frac{1}{2}(s+2)+2/3}{(s+2)^2+2}\right)$

$= \frac{1}{2}\mathcal{L}^{-1}\left(\dfrac{s+2}{(s+2)^2+(\sqrt{2})^2}\right) + \dfrac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left(\dfrac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}\right)$

$= \frac{1}{2}e^{-2t}\mathcal{L}^{-1}\left(\dfrac{s}{s^2+(\sqrt{2})^2}\right) + \dfrac{2}{3\sqrt{2}}e^{-2t}\mathcal{L}^{-1}\left(\dfrac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right)$

$= \frac{1}{2}e^{-2t}\cos(\sqrt{2}\,t) + \dfrac{2}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}\,t).$

Example: Solve the following IVP.

$$y'' - 6y' + 9y = t^2e^{3t}, \quad y(0)=2, \quad y'(0)=17$$

Using Laplace Transform.

sol: $\mathcal{L}(y''-6y'+9y) = \mathcal{L}(t^2e^{3t})$.

$$[s^2Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s) = \frac{2!}{(s-3)^3}$$

$$\Rightarrow [s^2-6s+9]Y(s) - 2s - 17 + 12 = \frac{2!}{(s-3)^3}$$

$$\therefore \quad (s-3)^2\, Y(s) = \frac{2!}{(s-3)^3} + 2s + 5$$

$$\Rightarrow \quad Y(s) = \frac{2!}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{2!}{(s-3)^5}\right) + \mathcal{L}^{-1}\left(\frac{2s+5}{(s-3)^2}\right)$$

example (3)

$$= \frac{2!}{4!}\,\mathcal{L}^{-1}\left(\frac{4!}{(s-3)^5}\right) + (2+11t)\,e^{3t}$$

$$= \frac{1}{12}\, e^{3t}\, t^4 + (2+11t)\,e^{3t}$$

$$= \left(\frac{1}{12}t^4 + 11t + 2\right) e^{3t}.$$

Home work :

Solve the following IVP Using Laplace Transform.

$$y'' + 4y' + 6y = 1 + e^{-t}$$

$$y(0) = y'(0) = 0.$$

Def: The unit step function or Heaviside function.

The unit step function or Heaviside function is defined by

$$u_c(t) = u(t-c) = \begin{cases} 0 & , t < c \\ 1 & , t \geqslant c \end{cases} \quad , c \geqslant 0$$

Example: $$u_5(t) = \begin{cases} 0 & , t < 5 \\ 1 & , t \geqslant 5 \end{cases}$$

Example: Sketch the graph of $y = u_3(t) - u_2(t)$

$$u_3(t) - u_2(t) = \begin{cases} 0 & , t < 3 \\ 1 & , t \geqslant 3 \end{cases} - \begin{cases} 0 & , t < 2 \\ 1 & , t \geqslant 2 \end{cases}$$

$$= \begin{cases} 0 - 0 & , 0 \leq t < 2 \\ 0 - 1 & , 2 \leq t < 3 \\ 1 - 1 & , t \geqslant 3 \end{cases}$$

$$= \begin{cases} 0 & , 0 \leq t < 2 \ \& \ t \geqslant 3 \\ -1 & , 2 \leq t < 3 \end{cases}$$

Remark : The Unit step function can be Used to write a piecewise function in a Compact form as follows :

$$f(t) = \begin{cases} g(t) & , 0 \le t < a \\ h(t) & , a \le t < b \\ k(t) & , t \geq b \end{cases}$$

then, $f(t) = g(t) + \big(h(t) - g(t)\big) u_a(t) + \big(k(t) - h(t)\big) u_b(t)$

Example: Write the following in a Compact form

① $f(t) = \begin{cases} \sin t & , 0 \le t < \frac{\pi}{4} \\ \sin t + \cos\left(t - \frac{\pi}{4}\right) & , t \geq \frac{\pi}{4} \end{cases}$

sol: $f(t) = \sin t + \cos\left(t - \frac{\pi}{4}\right) u_{\frac{\pi}{4}}(t)$

② $f(t) = \begin{cases} 20t & , 0 \le t < 5 \\ 2 & , 5 \le t < 9 \\ -1 & , t \geq 9 \end{cases}$

sol: $f(t) = 20t + (2 - 20t) u_5(t) - 3 u_9(t)$.

Example: Find $\mathcal{L}(u_c(t))$

Sol: $\mathcal{L}(u_c(t)) = \int_0^\infty u_c(t)\, e^{-st}\, dt$

$$= \int_c^\infty 1.\, e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_c^A e^{-st}\, dt = \lim_{A\to\infty} \frac{e^{-st}}{-s}\Big|_c^A$$

$$= \lim_{A\to\infty} \left(\frac{e^{-sA}}{-s} + \frac{e^{-sc}}{s} \right)$$

$$\mathcal{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Example: ① $\mathcal{L}(u_2(t)) = \dfrac{e^{-2s}}{s}$

② $\mathcal{L}^{-1}\left(\dfrac{e^{-6s}}{s}\right) = u_6(t)$

Theorem (6.3.1). Second Translation Theorem.

If $\mathcal{L}(f(t)) = F(s)$ and $a > 0$, then

$$\mathcal{L}(f(t-a)\,u_a(t)) = e^{-as} F(s) \quad , \quad s > a$$

OR: $$\mathcal{L}(f(t)\,u_a(t)) = e^{-as}\,\mathcal{L}(f(t+a)) \qquad t \longrightarrow t+a$$

and $$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\,u_a(t)$$

$$= \mathcal{L}^{-1}(F(s))\Big|_{t \to t-a} \cdot u_a(t).$$

Proof: $$\mathcal{L}(f(t-a)\,u_a(t)) = \int_0^\infty e^{-st} u_a(t)\, f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt = \lim_{A\to\infty} \int_a^A e^{-st} f(t-a)\, dt$$

$$\Rightarrow \lim_{A\to\infty} \int_0^A e^{-s(w+a)} f(w)\, dw$$

Let $w = t-a$
$dw = dt$
$t=a,\ w=0$
$t=\infty,\ w=\infty$

$$= \lim_{A\to\infty} \int_0^A e^{-sa} \cdot e^{-sw} f(w)\, dw = e^{-as} F(s)$$

Example: Find the following:

① $$\mathcal{L}\left(t^2 u_3(t)\right) = e^{-3s}\,\mathcal{L}(t^2)_{t\to t+3}$$

$$= e^{-3s}\,\mathcal{L}\left((t+3)^2\right)$$

$$= e^{-3s}\,\mathcal{L}\left(t^2+6t+9\right)$$

$$= e^{-3s}\left(\frac{2!}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right).$$

② $\mathcal{L}(f(t))$, where $f(t) = \begin{cases} 0 & , t<\pi \\ t-\pi & , \pi \le t < 2\pi \\ 0 & , t \geq 2\pi. \end{cases}$

First we write $f(t)$ in the compact form.

$$f(t) = 0 + (t-\pi-0)\,u_\pi(t) + \left(0-(t-\pi)\right)u_{2\pi}(t)$$

$$= (t-\pi)\,u_\pi(t) + (\pi-t)\,u_{2\pi}(t)$$

$$\therefore \mathcal{L}(f(t)) = \mathcal{L}\left((t-\pi)\,u_\pi(t)\right) + \mathcal{L}\left((\pi-t)\,u_{2\pi}(t)\right) \quad \ldots (*)$$

$$= e^{-\pi s}\,\mathcal{L}(t) + \mathcal{L}\left(\pi\,u_{2\pi}(t)\right) - \mathcal{L}\left(t\,u_{2\pi}(t)\right)$$

$$= e^{-\pi s} \cdot \frac{1}{s^2} + \pi \cdot \frac{e^{-2\pi s}}{s} - e^{-2\pi s} \mathcal{L}(t + 2\pi)$$

$$= e^{-\pi s} \cdot \frac{1}{s^2} + \pi \frac{e^{-2\pi s}}{s} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{2\pi}{s}\right)$$

$$= e^{-\pi s} \cdot \frac{1}{s^2} + e^{-2\pi s}\left(-\frac{1}{s^2} - \frac{\pi}{s}\right).$$

OR: Eq. (*) becomes:

$$\mathcal{L}(f(t)) = e^{-\pi s} \mathcal{L}((t+\pi)-\pi) + e^{-2\pi s} \mathcal{L}(\pi - (t+2\pi))$$

$$= e^{-\pi s} \mathcal{L}(t) + e^{-2\pi s} \mathcal{L}(-t - \pi)$$

$$= e^{-\pi s} \cdot \frac{1}{s^2} + e^{-2\pi s}\left(\frac{-1}{s^2} - \frac{\pi}{s}\right).$$

---

(3) $\mathcal{L}(u_3(t) \sin 4t) = e^{-3s} \mathcal{L}(\sin 4(t+3))$

$$= e^{-3s} \mathcal{L}(\sin(4t+12)) = e^{-3s} \mathcal{L}(\sin 4t \cos 12 + \sin 12 \cos 4t)$$

$$= e^{-3s} \cos 12 \left(\frac{4}{s^2+16}\right) + e^{-3s} \sin 12 \left(\frac{s}{s^2+16}\right)$$

Example: Find the following:

a) $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+4}\right) = u_2(t)\, \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)_{t \to t-2}$

$= \frac{1}{2} u_2(t) \sin 2(t-2).$

b) $\mathcal{L}^{-1}\left(\frac{e^{-4s}}{s}\right) = \mathcal{L}^{-1}\left(e^{-4s} \cdot \frac{1}{s}\right) = u_4(t) \cdot \mathcal{L}^{-1}\left(\frac{1}{s}\right)_{t \to t-4}$

$= u_4(t) \cdot 1 = u_4(t)$

In General:

$$\mathcal{L}^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t).$$

c) $\mathcal{L}^{-1}\left(\frac{1-e^{-2s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right)$

$= t - u_2(t)\, \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)_{t \to t-2}$

$= t - u_2(t)(t-2)$

d) $\mathcal{L}^{-1}\left(\frac{2(s-1)\, e^{-2s}}{s^2-2s+2}\right) = 2\,\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2+1}\, e^{-2s}\right)$

$= 2\, u_2(t)\, \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+1}\right)_{t \to t-2}$

$= 2\, u_2(t)\left(e^{t}\cos t\right)_{t \to t-2}$

$= 2\, u_2(t)\, e^{(t-2)}\, \cos(t-2).$

---

e) $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+9}\right) = u_1(t)\, \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)_{t \to t-1}$

$= u_1(t)\, \frac{1}{3}\left(\sin(3t)\right)_{t \to t-1} \qquad = \frac{1}{3}\, u_1(t) \sin 3(t-1).$

---

f) $\mathcal{L}^{-1}\left(\frac{(1-3s)\, e^{-2s}}{s^2+6s+25}\right) = \mathcal{L}^{-1}\left(\frac{(1-3s)\; e^{-2s}}{(s+3)^2+16}\right.$

$= \mathcal{L}^{-1}\left(\left(\frac{-3(s+3)+10}{(s+3)^2+16}\right) e^{-2s}\right) = \mathcal{L}^{-1}\left(\frac{-3(s+3)}{(s+3)^2+16} + \frac{10}{(s+3)^2+16}\right) u_2(t)$

$t \to t-2$

$= u_2(t)\left[-3\, e^{-3t}\cos 4t + \frac{10}{4} e^{-3t} \sin 4t\right]_{t \to t-2}$

$= u_2(t)\left[-3\, e^{-3(t-2)}\cos 4(t-2) + \frac{10}{4}\, e^{-3(t-2)} \sin 4(t-2)\right]$

6.4 Differential Equations with discontinuous Forcing functions

Example: Solve the following IVP Using Laplace Transform.

$$y'' + 4y = \sin t \; u_{2\pi}(t) \;,\; y(0) = 1 \;,\; y'(0) = 0.$$

Sol: $\mathcal{L}(y'' + 4y) = \mathcal{L}(\sin t \; u_{2\pi}(t))$

$$s^2 Y(s) - s\,y(0) - \cancel{y'(0)}^{\,0} + 4\,Y(s) = e^{-2\pi s}\, \mathcal{L}(\sin t)\Big|_{t \to t+2\pi}$$

$$(s^2+4)\,Y(s) = s + e^{-2\pi s} \cdot \mathcal{L}(\sin(t+2\pi))$$

$$= s + e^{-2\pi s}\, \mathcal{L}(\sin t)$$

periodic of period 2π.

$$\Rightarrow Y(s) = \frac{s}{(s^2+4)} + e^{-2\pi s} \cdot \frac{1}{(s^2+1)(s^2+4)}$$

$$\therefore\; y(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right)\Bigg|_{t \to t-2\pi} \cdot u_{2\pi}(t)$$

$$y(t) = \cos 2t + u_{2\pi}(t)\, \mathcal{L}^{-1}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}\right)\Bigg|_{t \to t-2\pi}$$

Now, $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$\Rightarrow 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$

$$\Rightarrow \left.\begin{aligned} 0 &= A + C && (\text{coef. } s^3) \\ 0 &= B + D && (\text{coef. } s^2) \\ 0 &= 4A + C && (\text{coef. } s^1) \\ 1 &= 4B + D && (\text{coef of } s^0) \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 0 \\ B &= \tfrac{1}{3} \\ C &= 0 \\ D &= -\tfrac{1}{3}. \end{aligned}$$

$$\therefore y(t) = \cos 2t + u_{2\pi}(t)\, \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4} \right)_{t \to t-2\pi}$$

$$= \cos 2t + \frac{1}{3} u_{2\pi}(t) \sin(t-2\pi) - \frac{1}{6} u_{2\pi}(t) \sin(2(t-2\pi))$$

$$= \cos 2t + u_{2\pi}(t) \left[\frac{1}{3} \sin t - \frac{1}{6} \sin 2t \right]$$

$$= \begin{cases} \cos 2t, & 0 \le t < 2\pi \\ \cos 2t + \frac{1}{3}\sin t - \frac{1}{6}\sin 2t, & t \ge 2\pi. \end{cases}$$

Example: Solve $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 2$,

where $f(t) = \begin{cases} \frac{1}{2} t & , 0 \le t < 6 \\ 3 & , t \geq 6 . \end{cases}$

Sol: $f(t) = \frac{1}{2} t + \left(3 - \frac{1}{2} t\right) u_6(t)$

$$\mathcal{L}(f(t)) = \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(3 - \tfrac{1}{2}t\right)_{t \to t+6} .$$

$$= \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(3 - \tfrac{1}{2}(t+6)\right)$$

$$= \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(-\tfrac{1}{2}t\right)$$

$$= \frac{1}{2s^2} + e^{-6s}\left(\frac{-1}{2s^2}\right) = \frac{1 - e^{-6s}}{2s^2}$$

Now: $\mathcal{L}(y'' + y) = \mathcal{L}(f(t))$

$$\Rightarrow s^2 Y(s) - \cancel{s\,y(0)} - y'(0) + Y(s) = \frac{1 - e^{-6s}}{2s^2}$$

$$\Rightarrow (s^2 + 1) Y(s) = \frac{1 - e^{-6s}}{2s^2} + 2$$

$$\therefore Y(s) = \frac{2}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}$$

(**)

$$\Rightarrow y(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right) - u_6(t)\mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right)$$

$t \to t-6$

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}\right]$$

$$\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2.$$

$$1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\Rightarrow \left.\begin{array}{ll} 0 = A + C & (s^3) \\ 0 = B + D & (s^2) \\ 0 = A & (s^1) \\ 1 = B & (s^0) \end{array}\right\} \Rightarrow \begin{array}{l} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{array}$$

$$\therefore \frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{0}{s} + \frac{1}{s^2} + \frac{-1}{s^2+1}\right]$$

Back to (**) :

$$\mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right) = \frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left(t - \sin t\right)$$

$$\therefore \; y(t) = 2\sin t + \frac{1}{2}\left(t - \sin t\right) - \frac{1}{2}u_6(t)\left(t - \sin t\right)\Big|_{t \to t-6}$$

$$= 2\sin t + \frac{1}{2}t - \frac{1}{2}\sin t - \frac{1}{2}u_6(t)\Big((t-6) + \sin(t-6)\Big)$$

$$= \frac{3}{2}\sin t + \frac{1}{2}t - \frac{1}{2}u_6(t)\left(t - 6 + \sin(t-6)\right)$$

Home work: Solve Using Laplace Transform the IVP:

① $2y'' + y' + 2y = g(t)$, $y(0) = y'(0) = 0$, where

$$g(t) = \begin{cases} 1 & , \; 5 \le t < 20 \\ 0 & , \; 0 \le t < 5 \;\&\; t \ge 20 \end{cases} = u_5(t) - u_{20}(t)$$

② $y'' + 4y' + 3y = h(t)$, $y(0) = y'(0) = 0$, where

$$h(t) = \begin{cases} 0 & , \; 0 \le t < 2 \\ -1 & , \; t \ge 2. \end{cases}$$

Good Luck

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