Birzeit University

Mathematics Department

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Course Code: MATH331

Title: Ordinary Differential Equations

Chapter 1 Introduction

Sections 1.3, 1.1, 1.2

1.3 Classification and Differential Equations

- Differential equations are relation containing derivatives.

DEs
- ODEs (Ordinary differential equations)
- PDEs (Partial differential equations)
- System of DEs

[1] Ordinary Differential Equations (ODEs).

The unknown function depends on one independent variable and only ordinary derivatives appear in the equation.

Ex. (a) $\dfrac{dv}{dt} = 9.8 - \frac{1}{5}v$ (the unknown function is $v = v(t)$: v dependent variable

t : independent variable.

(b) $\frac{dp}{dt} = \frac{1}{2}p - 450$ is ode ($p = p(t)$ is the unknown function)

(c) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + y = x^2$

is ode with the unknown function $y = y(x)$.

2 Partial Differential Equations (PDEs)

the unknown function depends on two or more independent variables and partial derivatives appear in the equation.

ex. (i) $\alpha^2 u_{xx} = u_t$ or $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

is called heat equation. (unknown function $u = u(x,t)$).

(ii) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ (wave equation).

3 System of Differential Equations

Two or more unknown functions require a system of Differential equations.

ex: (Lotka - Volterra) equations

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy \\ \dfrac{dy}{dt} = -cy + \gamma xy. \end{cases}$$

Unknowns ($x = x(t)$, $y = y(t)$).

• <u>the order of a D.E</u> is the order of the heighest derivative that appears in the equation.

ex: (1) $\dfrac{dy}{dt} - ty = t^3$ (first order).

(2) $\left(\dfrac{d^2q}{dx^2}\right)^5 + \cos(x+q) = 0$ (2^{nd} order).

Linear and Nonlinear DEs

The ode $\boxed{F(t, y, y', ---, y^{(n)}) = 0}$ (*)

is said to be linear if F is a linear function of the variables $y, y', ---, y^{(n)}$.

Thus, the general linear ode of order n is $\boxed{a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + --- + a_n(t)\, y = g(t)}$ (**)

An equation that is not of the form (**) is a nonlinear equation.

Ex. Determine the order of the following de's and state whether the equation is linear or nonlinear.

$\boxed{1}$ $y' - 2y = t^3$

1st order linear ode.

(2) $t^2 y'' + t y' + (\sin t) y = 0$

2nd order ode (linear in y).

(3) $\frac{dp}{dt} + t p^2 = \cos t$.

first order ode (non linear).

(4) $\frac{d^2 q}{dx^2} + \cos(x + q) = 0$

2nd order ode (non linear).

(5) $\frac{d^3 x}{dy^3} + \left(\frac{d^2 x}{dy^2}\right)^5 + y^6 = x$.

third order <u>nonlinear</u> ode.

(6) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial y} = x^2 + y^2$

third order linear PDE.

(7) $(x + e^y)\, dy = dx$ ode

$\frac{dy}{dx} = \frac{1}{x + e^y}$ nonlinear in y.

But $\frac{dx}{dy} = x + e^y$ linear in x.

<u>Solutions</u> A solution of the ode (*), on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', ---, \phi^{(n)}$ exist and satisfy $F(t, \phi, \phi', --, \phi^{(n)}) = 0$ for every t in (α, β).

<u>Ex.</u> verify that $\boxed{y = 3x + x^2}$ is a solution of the d.e $\boxed{x \frac{dy}{dx} - y = x^2}$.

<u>Sol.</u> $\frac{dy}{dx} = 3 + 2x$.

$$L.H.S = x\frac{dy}{dx} - y = x(3+2x) - (3x + x^2)$$
$$= 3x + 2x^2 - 3x - x^2$$
$$= x^2 = R.H.S$$

Ex. verify that $\boxed{y = (\cos t)\ln(\cos t) + t\sin t}$

is a solution of the ode

$$\boxed{y'' + y = \sec t, \quad 0 < t < \frac{\pi}{2}}$$

Sol. $y' = -\sin t \ln(\cos t) + \cos t\left(-\frac{\sin t}{\cos t}\right)$

$\qquad + \sin t + t\cos t$

$$\boxed{y' = -\sin t \ln(\cos t) + t\cos t}$$

$y'' = -\cos t \ln(\cos t) - \sin t\left(-\frac{\sin t}{\cos t}\right)$

$\qquad + \cos t - t\sin t$

$$\boxed{\begin{aligned} y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t \\ - t\sin t \end{aligned}}$$

L.H.S (ode) $= y'' + y$

$$= \frac{\sin^2 t}{\cos t} + \cos t = \frac{\sin^2 t + \cos^2 t}{\cos t}$$

$$= \frac{1}{\cos t} = \sec t = \text{R.H.S.}$$

(H.Ws) 1 verify that

$$\boxed{y = e^{t^2}\int_0^t e^{-r^2}\,dr + e^{t^2}}$$ is a solution

of $\boxed{y' - 2ty = 1}$.

2 verify that $\boxed{y = \frac{\ln x}{x^2}}$ is a solution

of $\boxed{x^2 y'' + 5xy' + 4y = 0, \quad x > 0.}$

1.1 Some basic Models and Direction fields

Ex1. Suppose that an object is falling in the atmospher near sea level. Formulate a differential equation that describes the motion.

Solution: Let us use t to denote time (Independent variable).

v represent the velocity of the falling object (dependent variable).

Using Newton's Second Law, which states $\boxed{F = ma}$ — (1)

where F: the net force exerted on the object.

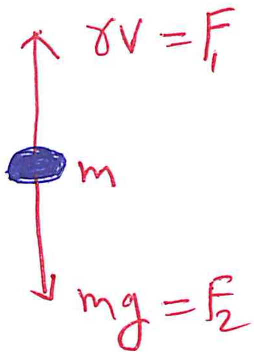
m: mass of the object.

a: acceleration.

We can rewrite Eq(1) as

$$F_{net} = F_2 - F_1, \quad F_1 : \text{drag force}$$

$$ma = mg - \gamma v$$



$$m\frac{dv}{dt} = mg - \gamma v$$

OR $$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m} v} \quad (2)$$

where g: the acceleration due to gravity.

γ: drag coefficient.

v: velocity.

Eq(2) is a D.E (1st order linear d.e).

Rmk. To solve eq(2), we need to find a function $v = v(t)$ that satisfies the equation (next section) (Section 1.2).

Our task: Investigate the behavior of the solution for D.E ② without solving it. this is called direction field or slope field.

To do that take, for example, $m = 10kg$, $\gamma = 2kg/s$. In this case, eq ② becomes

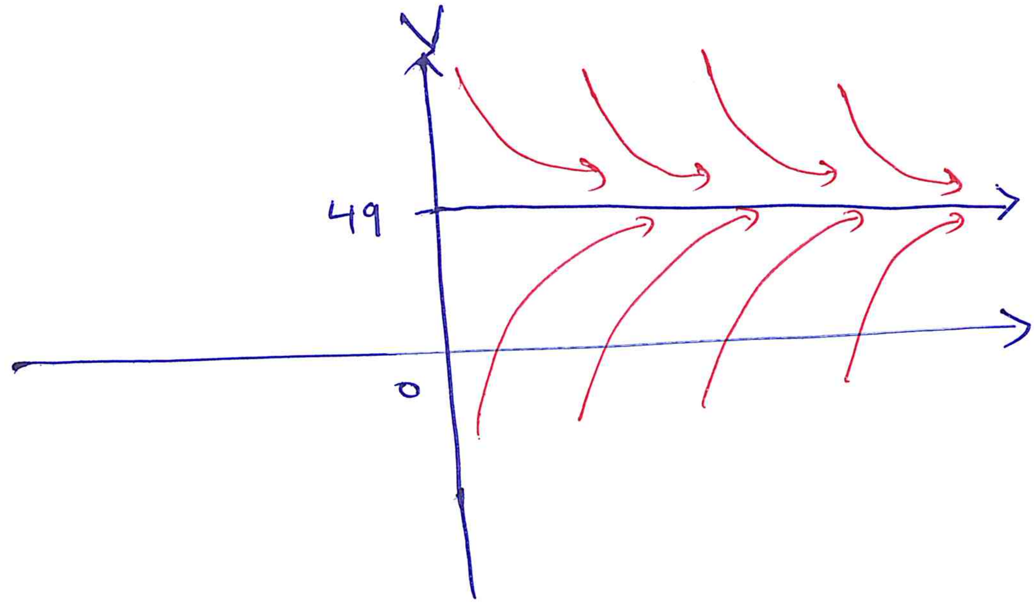
$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}} \quad (3)$$

Now, we find the equilibrium solution of the D.E (3) by setting $\frac{dv}{dt} = 0$. This implies $9.8 - \frac{v}{5} = 0 \Rightarrow \boxed{v = 49}$

Next, we choose values for v below 49, take $v = 5 \Rightarrow \frac{dv}{dt} = 9.8 - 1 = 8.8 > 0$.

Then choose values for $v > 49$ (take $v_0 = 80$)

$$\Rightarrow \frac{dv}{dt} = 9.8 - 16 < 0$$



(A direction field and equilibrium solution for eq(3)).

Rmk. Solutions below the equilibrium solution ($v=49$) increase with time, those above it decrease with time and all other solutions approach ($v=49$). That is $\boxed{\lim_{t\to\infty} v(t) = 49.}$

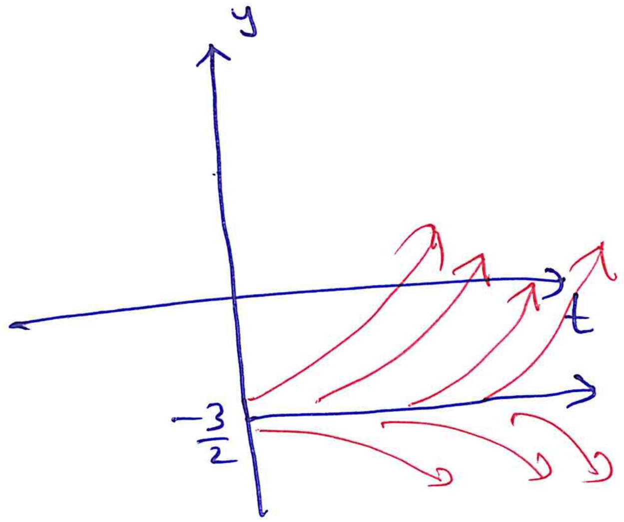
Ex. Draw a direction field for the given d.e, determine the behavior of y as $t \to \infty$.

$\boxed{1}$ $\dfrac{dy}{dt} = 2y+3.$

- $\dfrac{dy}{dt} = 0 \Rightarrow 2y+3=0 \Rightarrow \boxed{y=-3/2}$ is the equilibrium solution.

- If $y_0 > -\frac{3}{2} \Rightarrow \dfrac{dy}{dt} > 0$

  (for ex., take $y_0 = 0 \Rightarrow$
  $\dfrac{dy}{dt} = 3 > 0$).



- If $y_0 < -\frac{3}{2} \Rightarrow \dfrac{dy}{dt} < 0$

  (take $y_0 = -2 \Rightarrow \dfrac{dy}{dt} = -4+3 < 0$)

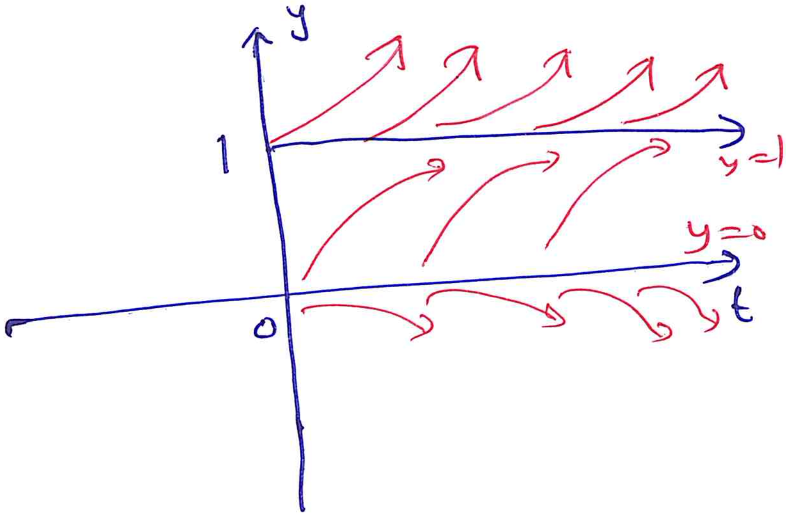
<u>Behavior</u>. $\lim\limits_{t\to\infty} y(t) = \begin{cases} +\infty & \text{if } y_0 > -\frac{3}{2} \\ -\infty & \text{if } y_0 < -\frac{3}{2} \end{cases}$

Therefore, the solution <u>diverges</u> from $-\frac{3}{2}$ as $t \to \infty$.

$\boxed{2}$ $y' = y(y-1)^2$

- $\frac{dy}{dt} = 0 \Rightarrow y = 0$ or $(y-1)^2 = 0$

  $\boxed{y=0}$ or $\boxed{y=1}$ are the equilibrium solutions.



- If $y_0 < 0 \Rightarrow \boxed{\frac{dy}{dt} < 0}$

If y_0 is between 0 and 1, then

$\boxed{\frac{dy}{dt} > 0}$

- If $y_0 > 1$, then $\boxed{\frac{dy}{dt} > 0}$

<u>Behavior</u>. If the initial value is negative ($y_0 < 0$), then y diverges from 0.

If the initial value is between 0 and 1, then $y \to 1$ as $t \to \infty$.

If the intial value is greater than 1, then the Solution y diverges from 1 as $t \to \infty$

$\boxed{3}$ $y' = y(y-1)^2$, $y(0) = 2020$.

from ex 2, y diverges

4 $\begin{cases} y' = y(y-1)^2 \\ y(0) = 0.1 \end{cases}$

From ex 2, $\lim_{t\to\infty} y(t) = 1$.

Rmk. A d.e together with initial condition is called initial value problem (IVP). like ex 3 and ex 4.

Example. (Field mice and owls)

Consider a population of field mice who inhabit a certain rural area. Assume that the mouse population increases at a rate proportional to the current population. The DE that describes the growth $p(t)$ is

$$\boxed{\frac{dp}{dt} = rp} \quad \cdots (4)$$

where r: is called the rate constant or growth rate.

p: the population of mice field.

t: time.

Ex. Assume that $r = 0.5$/month and owls are present and they kill 15 field mice per day. So, the D.E (4) becomes $\boxed{\frac{dp}{dt} = \frac{1}{2}p - 450}$ — (5)

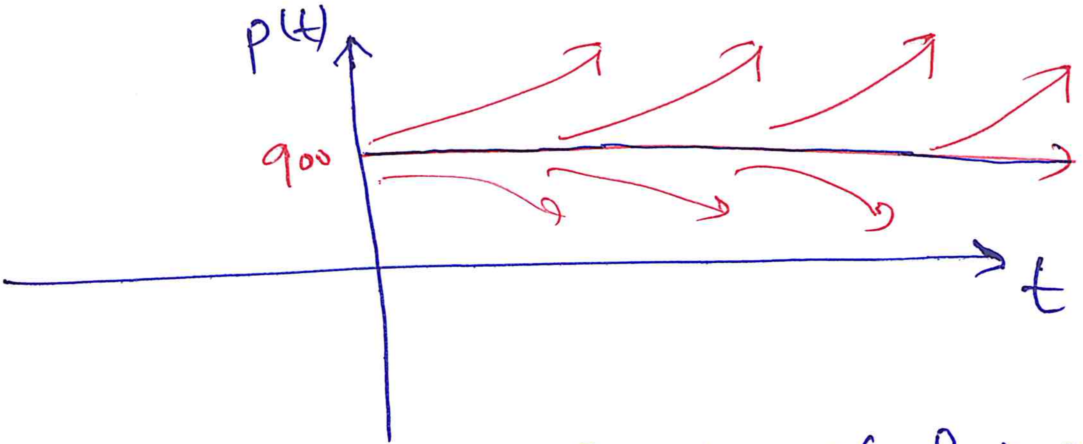
Now, we will study the behavior of the solution for D.E (5) without solving it.

• The equilibrium solution of Eq (5)

$$\frac{dp}{dt} = 0 \Rightarrow \frac{1}{2}p - 450 = 0 \Rightarrow \boxed{p = 900}$$

If $P_0 < 900$ (take $p_0 = 0$), then $\frac{dp}{dt} < 0$

If $P_0 > 900$, then $\frac{dp}{dt} > 0$



Behavior $\lim_{t\to\infty} p(t) = \begin{cases} +\infty, & \text{if } P_0 > 900 \\ 0, & \text{if } P_0 < 900. \end{cases}$

Not $-\infty$ since p(t) is a population.

1.2 Solutions of Some Differential Equations

Recall, In section 1.1, we derived the d.e's

$$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m} v} \quad \text{— (1) (Falling object)}$$

and

$$\boxed{\frac{dp}{dt} = rp - k} \quad \text{— (2) (Population of field mice and owls).}$$

Both DE's (1) and (2) are of the general form:

$$\boxed{\frac{dy}{dt} = ay - b} \quad \text{— (3)}$$

where a, b are constants.

Aim. we need to find the exact solution of (1) and (2) for a given m, g, γ, r and k as follows.

Ex (1) Solve $$\boxed{\frac{dp}{dt} = \frac{1}{2}p - 450} \quad (4)$$

Solution. Rewrite (4) in the form

$$\frac{dp}{dt} = \frac{p-900}{2}$$

or if $p \neq 900$, $\boxed{\frac{dp}{p-900} = \frac{1}{2}\,dt}$ — (5)

then, by integrating both sides of (5), we get $\int \frac{dp}{p-900} = \int \frac{1}{2}\,dt$

$$\Rightarrow \ln|p-900| = \frac{1}{2}t + C$$

$$\Rightarrow |p-900| = e^{\frac{1}{2}t+C} = e^{C}\cdot e^{\frac{1}{2}t}$$

$$\Rightarrow p-900 = \pm e^{C} e^{\frac{1}{2}t}$$

$$\Rightarrow \boxed{p(t) = 900 + A e^{\frac{1}{2}t}}$$, where $A = \pm e^{C}$ is a nonzero constant.

Ex② Solve the IVP

$$\begin{cases} \frac{dp}{dt} = \frac{1}{2}p - 450 \\ p(0) = 850 \end{cases}$$

Solution. In example (1), we found

$$p(t) = 900 + Ae^{\frac{1}{2}t}$$

Now, $p(0) = 900 + A = 850 \Rightarrow \boxed{A = -50}$

$$\therefore\ p(t) = 900 - 50e^{\frac{1}{2}t}$$

Notice that $\lim_{t\to\infty} p(t) = 0$ (Not $-\infty$ because p is a population).

Ex 3. Solve the IVP

$$\begin{cases} \dfrac{dv}{dt} = 9.8 - \frac{1}{5}v \\ v(0) = 0 \end{cases}$$

Sol: If $v \neq 49$, $\dfrac{dv}{9.8 - \frac{1}{5}v} = dt$

$$\Rightarrow -5\int \frac{-\frac{1}{5}\,dv}{9.8 - \frac{1}{5}v} = \int dt$$

$$-5\,\ln\left|9.8 - \tfrac{1}{5}v\right| = t + C_1$$

$$\Rightarrow \ln\left|9.8 - \tfrac{1}{5}v\right| = -\frac{t}{5} + C_2$$

$$|9.8 - \tfrac{1}{5}v| = e^{C_2} e^{-\frac{1}{5}t}$$

$$\Rightarrow 9.8 - \tfrac{1}{5}v = \underbrace{\pm e^{C_2}}_{C_3} e^{-\frac{1}{5}t}$$

$$\Rightarrow \tfrac{1}{5}v = 9.8 - C_3 e^{-\frac{1}{5}t}$$

$$\Rightarrow v(t) = 49 - 5C_3 e^{-\frac{1}{5}t}$$

$$\boxed{v(t) = 49 + B e^{-\frac{1}{5}t}}, \quad B = -5C_3$$

$$v(0) = 49 + B = 0 \Rightarrow \boxed{B = -49}$$

Finally, $\boxed{v(t) = 49 - 49 e^{-\frac{1}{5}t}}$

Notice that, $\lim_{t\to\infty} v(t) = 49$. (Behavior).

Ex4.

Solve the IVP

$$\begin{cases} \dfrac{dy}{dt} = ay - b \\ y(0) = \alpha. \end{cases}$$

If $y \neq \frac{b}{a}$, $a \neq 0$, we have

$$\int \frac{dy}{ay-b} = \int dt$$

$$\Rightarrow \ln|ay-b| = at + C_1$$

$$\Rightarrow ay - b = \pm e^{C_1} e^{at} = Ae^{at}$$

$$\Rightarrow ay = b + Ae^{at}$$

$$\boxed{y = \frac{b}{a} + Be^{at}} \text{ where } B = \frac{A}{a}.$$

$$\alpha = y(0) = \frac{b}{a} + B \Rightarrow \boxed{B = \alpha - \frac{b}{a}}$$

Finally, $\boxed{y_{(t)} = \frac{b}{a} + \left(\alpha - \frac{b}{a}\right)e^{at}}$

- <u>Some Important questions dealing with DEs</u>

(1) Is there a solution of the d.e ? (Existence)

(2) If the solution exists, is it unique ? (Uniquence).

(3) How to find the solution if it exists ?.

Chapter 2 First Order Differential Equations

2.2 seperable Equations

The general form of the first order d.e is

$$\boxed{\frac{dy}{dx} = f(x,y)} \quad (1)$$

we can rewrite eq(1) in the form

$$\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0} \quad (2)$$

by setting $M(x,y) = -f(x,y)$ and $N(x,y) = 1$.

If M is a function of x only and N is a function of y only, then eq(2) becomes

$$\boxed{M(x) + N(y)\frac{dy}{dx} = 0} \quad (3)$$

Such an eq. is said to be seperable, because if it is written in the form $M(x)\,dx + N(y)\,dy = 0$, we can solve it by integrating M and N.

Thus, the general form of a first order seperable d.e is $\boxed{\dfrac{dy}{dx} = g(x)\, h(y).}$

Ex ①. Solve the IVP

$$\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)}, \quad y(0) = -1.$$

Solution. $\int 2(y-1)\, dy = \int (3t^2+4t+2)\, dt$.

$$(y-1)^2 = t^3 + 2t^2 + 2t + C$$

$y(0) = -1$: $(-1-1)^2 = 0 + C \Rightarrow \boxed{C = 4}$

$$\therefore (y-1)^2 = t^3 + 2t^2 + 2t + 4$$

$$y = 1 \pm \sqrt{t^3+2t^2+2t+4}$$

Now, $y(0) = 1 \pm \sqrt{4} = 1 \pm 2 = 3$ or -1 ✓

$$\therefore \boxed{y = 1 - \sqrt{t^3+2t^2+2t+4}}$$

$$t^3+2t^2+2t+4 \geq 0$$

$$t^2(t+2)+2(t+2) \geq 0$$

$$(t^2+2)(t+2) \geq 0$$

$$\Rightarrow t \geq -2$$

Finally, $y = 1 - \sqrt{t^3+2t^2+2t+4}$, $t \geq -2$

is called the <u>explicit solution</u> of our IVP.

<u>Ex 2.</u> Solve the IVP

$$\begin{cases} x\, e^{2x+\cos y} + (\sin y)\, y' = 0 \\ y(0) = \frac{\pi}{2} \end{cases}$$

<u>Solution.</u> $x\, e^{2x} \cdot e^{\cos y} + (\sin y) \dfrac{dy}{dx} = 0$

$$\Rightarrow x e^{2x}\, dx + (\sin y)\, e^{-\cos y}\, dy = 0$$

$$\Rightarrow \boxed{\int x e^{2x}\,dx + \int (\sin y)\, e^{-\cos y}\,dy = 0} \quad (*)$$

- $\int x e^{2x}\,dx$

$u = x \qquad dv = e^{2x}\,dx$

$du = dx \qquad v = \frac{e^{2x}}{2}$

$= \frac{1}{2} x e^{2x} - \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c_1$ — (I)

- $\int (\sin y)\, e^{-\cos y}\,dy$

$u = -\cos y$

$du = +\sin y\,dy$

$= \int e^{u}\,du = e^{u} + c_2$

$= e^{-\cos y} + c_2$ — (II)

(I) and (II) into (*):

$$\boxed{\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} = -e^{-\cos y} + c}$$

$y(0) = \frac{\pi}{2}$: $\quad 0 - \frac{1}{4} = -e^{-\cos\frac{\pi}{2}} + c$

$$-\frac{1}{4} = -1 + c$$

$$\Rightarrow \boxed{c = \frac{3}{4}}$$

Finally, $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} = -e^{-\cos y} + \frac{3}{4}$

is the implicit solution of our problem.

Ex. (H.Ws) Solve

① $\frac{dy}{dx} = \frac{xy - 3x - y + 3}{xy - 2x + 4y - 8}$

② $\begin{cases} (x - xy^2) + (8y - x^2y)y' = 0 \\ y(2) = 2. \end{cases}$

Homogeneous D. E's

The general form of a homog. D. E's

is $\boxed{\dfrac{dy}{dx} = f(x,y) = F\left(\dfrac{y}{x}\right).}$ — (a)

Let $\dfrac{y}{x} = V$ or $\boxed{y = Vx}$ — (b)

$$\boxed{\frac{dy}{dx} = V + x\frac{dV}{dx}} \text{ — (c)}$$

Setting (b) and (c) into (a):

$V + x\dfrac{dV}{dx} = F(V)$ which is

a seperable d.e.

Ex. show that the following d.e is homog. and solve it.

$$\begin{cases} y' = \dfrac{3y^2 - x^2}{2xy} \\ y(1) = 2. \end{cases}$$

Write y as a function of x. Find the interval where the solution is defined.

Sol. $$\frac{dy}{dx} = \frac{3y^2}{2xy} - \frac{x^2}{2xy}$$

$$\boxed{\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{x}{y}\right) = F\left(\frac{y}{x}\right)} \quad (*)$$

$\therefore$ the d.e is homogeneous.

let $\frac{y}{x} = v$ or $\boxed{y = vx}$ — (1)

$\boxed{y' = v + x\frac{dv}{dx}}$ — (2)

setting (1) and (2) into (*):

$$v + x\frac{dv}{dx} = \frac{3}{2}v - \frac{1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{2}v - \frac{1}{2v} = \frac{v^2-1}{2v}$$

$$\Rightarrow \int \frac{2v}{v^2-1}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \ln|v^2-1| = \ln|x| + c$$

$$\boxed{\ln\left|\frac{y^2}{x^2}-1\right| = \ln|x| + c}$$

$y(1)=2: \ \ln|4-1| = \ln 1 + c$

$\Rightarrow \boxed{c=\ln 3}$

$$\therefore \boxed{\ln\left|\frac{y^2}{x^2}-1\right| = \underbrace{\ln|x| + \ln 3}_{\ln|3x|}} \quad \text{(the implicit solution)}$$

$$\Rightarrow \left|\frac{y^2}{x^2}-1\right| = |3x|$$

$$\frac{y^2}{x^2}-1 = 3x \quad \text{or} \quad \underbrace{\frac{y^2}{x^2}-1 = -3x}_{\text{reject since } y(1)=2}$$

$$\Rightarrow y^2 = x^2(3x+1)$$

$$y = \pm|x|\sqrt{3x+1}$$

Since $y(1)=2$, then $\boxed{y = x\sqrt{3x+1}}$

$$3x+1 \geq 0 \Rightarrow x \geq -\tfrac{1}{3}$$

$\therefore y = x\sqrt{3x+1}, \ x \geq -\frac{1}{3}$ is the explicit solution.

Ex. Solve $x\,dy = \left(x e^{\frac{y}{x}} + y + x\right) dx$

Sol. $\boxed{\dfrac{dy}{dx} = e^{\frac{y}{x}} + \dfrac{y}{x} + 1 = F\left(\dfrac{y}{x}\right)}$ (*)

let $\dfrac{y}{x} = v \Rightarrow y = vx$ — (i)

$y' = v + x\dfrac{dv}{dx}$ — (ii)

(i) + (ii) into (*):

$$v + x\frac{dv}{dx} = e^{v} + v + 1$$

$$\Rightarrow x\frac{dv}{dx} = e^{v} + 1$$

$$\Rightarrow \int \frac{1}{1+e^{v}}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow -\int \frac{-e^{-v}}{1+e^{-v}}\,dv = \int \frac{dx}{x}$$

$$-\ln\left(1+e^{-v}\right) = \ln|x| + c$$

$$-\ln\left(1+e^{-\frac{y}{x}}\right) = \ln|x| + c$$ is an implicit solution.

Ex. (H.W) (1) Solve the d.e

$$x\frac{dy}{dx} = y\ln\left(\frac{y}{x}\right), \quad x>0.$$

write y as a function of x.

(2) Solve the d.e

$$(y\ln y - y\ln x + y)\,dx = x\,dy.$$

(3) Solve the d.e

$$\left[x^2\sin\left(\frac{y^2}{x^2}\right) - 2y^2\cos\left(\frac{y^2}{x^2}\right)\right]dx$$

$$+\ 2xy\cos\left(\frac{y^2}{x^2}\right)dy = 0$$

2.1 linear Equations, Method of Integrating Factors.

Recall, that the general form of a first order d.e is $\boxed{\frac{dy}{dt} = f(t,y)}$ — (1)

where f is a given function of two variables. If the function f in Eq (1) depends linearly on y, then eq (1) is called a first order linear d.e. If f is not linear in y, then eq (1) will be nonlinear.

Thus, the general first order linear ode has the form

$$\boxed{\frac{dy}{dt} + p(t)y = g(t)} \quad \text{— (2)}$$

where p and g are given functions of t.

Ex. $\frac{dy}{dx} + xy = \sin x$ is linear in y.

ex. $\frac{d\beta}{d\alpha} + \alpha\beta^2 = \tan\alpha$ is nonlinear.

Rmk. If $p(t)$ and $g(t)$ are constants, we learned how to solve it (section 1.2).
what about if p and g are functions of t?

Ans. we use the method of integrating factor as follows.

Multiply both sides of eq(2) by a positive function $\mu(t)$:

$$\boxed{\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)} \quad (3)$$

Let us try to find $\mu(t)$ so that the LHS of eq(3) is the derivative of $\mu(t)y$.
that is, comparing the L.H.S of eq(3) with

$$\boxed{\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y} \quad (4)$$

We observe that $\boxed{\dfrac{d\mu(t)}{dt} = p(t)\,\mu(t)}$

$$\Rightarrow \int \frac{d\mu(t)}{\mu(t)} = \int p(t)\,dt$$

$$\Rightarrow \ln|\mu(t)| = \int p(t)\,dt + c$$

$$\Rightarrow \mu(t) = A\,e^{\int p(t)\,dt} \quad (\text{Take } A=1).$$

$$\therefore \boxed{\mu(t) = e^{\int p(t)\,dt}}$$

is called the integrating factor of eq (2).

Now, Back to eq. (2), multiply it by

$\mu(t) = e^{\int p(t)\,dt}$ and obtain

$$\boxed{e^{\int p(t)\,dt}\,\frac{dy}{dt} + p(t)\,e^{\int p(t)\,dt}\,y = g(t)\,\mu(t)}$$

$$\Rightarrow \frac{d}{dt}\left[\mu(t)\,y(t)\right] = g(t)\,\mu(t)$$

$$\Rightarrow \mu(t)\,y(t) = \int g(t)\,\mu(t)\,dt + C$$

Finally, $$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int g(t)\,\mu(t)\,dt + c\right]},$$

where $\boxed{\mu(t) = e^{\int p(t)\,dt}}$ is a solution of eq (2).

Summary.

the general solution of the first order linear d.e $\frac{dy}{dt} + p(t)y = g(t)$ is

$$y = \frac{1}{\mu(t)}\left[\int g(t)\,\mu(t)\,dt + c\right], \text{ where}$$

$\mu(t) = e^{\int p(t)\,dt}$ is the integrating factor.

Ex(1) Solve the IVP

$$\begin{cases} t\frac{dy}{dt} + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

write y as a function of t. Find the interval in which the solution is certain to exist.

(37)

Solution. $\dfrac{dy}{dt} + \underbrace{\dfrac{2}{t}}_{p(t)} y = \underbrace{4t}_{g(t)}$ (standard) ; $t \neq$

- $\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = t^2, \ t \neq 0$

- $y = \dfrac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]$

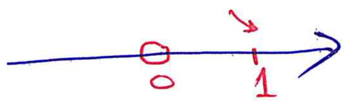
$= \dfrac{1}{t^2}\left[\int t^2 \cdot 4t\, dt + c\right]$

$y = \dfrac{1}{t^2}\left[t^4 + c\right] = t^2 + \dfrac{c}{t^2}, \ t \neq 0$

- $2 = y(1) = 1 + c \Rightarrow \boxed{c = 1}$

$\therefore y = t^2 + \dfrac{1}{t^2}, \ t \neq 0$ is the explicit solution.

- The largest interval in which the solution is certain to exist is $(0, \infty)$



Ex 2. Solve $\dfrac{dy}{dx} = \dfrac{y}{ye^y - 2x}$

the eq. is not linear in y but it is linear in x :

$$\frac{dx}{dy} = \frac{ye^y - 2x}{y} = \frac{\cancel{y}e^y}{\cancel{y}} - \frac{2}{y}x$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{2}{y}x = e^y} \quad \text{lin. in } x.$$

$$p(y) = \frac{2}{y}, \quad g(y) = e^y$$

- $$\mu(y) = e^{\int p(y)\,dy} = e^{\int \frac{2}{y}\,dy} = e^{2\ln|y|} = y^2, \; y>0$$

$$x(y) = \frac{1}{\mu(y)}\left[\int \mu(y)\, g(y)\,dy + c\right]$$

$$\boxed{x(y) = \frac{1}{y^2}\left[\int y^2 e^y\,dy + c\right]} \quad (*)$$

$$\int y^2 e^y \, dy$$

$$= y^2 e^y - 2y e^y + 2e^y + c$$

Derv.		Integration
y^2		e^y
	$(+)$	e^y
$2y$	$(-)$	e^y
2	$(+)$	e^y
0		

Back to $(*)$:

$$x(y) = \frac{1}{y^2}\left[y^2 e^y - 2y e^y + 2e^y + c \right]$$

$$\boxed{x = e^y - \frac{2e^y}{y} + \frac{2}{y^2} e^y + \frac{c}{y^2}}$$

Ex3. (H.W) Solve the IVP

$$(\mathrm{I}) \begin{cases} (\sin y)\, dx + 2(x - 3\sin y)\cos y \, dy = 0 \\ x\left(\frac{\pi}{2}\right) = \frac{1}{2} \end{cases}$$

Ans. $x(y) = 2\sin y - \frac{3}{2}\csc^2 y , \; 0 < y < \pi .$

$$(\mathrm{II}) \begin{cases} x \frac{dy}{dx} + xy = 1 - y \\ y(1) = 1 \end{cases}$$

2.3 Modeling with first order equations

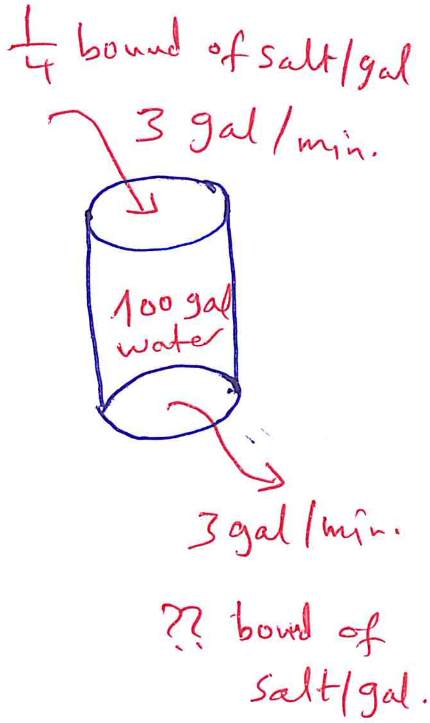
Ex 1. At time $t=0$ tank contains 50 bound of salt dissolved in 100 gal of water.
Assume that water containing $\frac{1}{4}$ bound of salt / gal is entering the tank at a rate of 3 gal / min. and leave it at the same rate.

(i) Set up the IVP that describes this process.

(ii) Find the amount of salt $Q(t)$ in the tank at any time t.

(iii) Find the limiting amount of salt Q_L in the tank after a very long time.

(iv) Find the time T when $Q(t) = 25.5$.



Sol: Let $Q(t)$ be the amount of salt in the tank at any time t, $Q(0) = 50$ bound

(i)

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= (\text{Concentration} \times \text{flow in}) - (\text{Concentration} \times \text{flow out})$$

$$= \left(\frac{1}{4}\right)(3) - \left(\frac{Q}{100}\right)(3)$$

So, the IVP is

$$\begin{cases} \dfrac{dQ}{dt} + \dfrac{3}{100} Q = \dfrac{3}{4} \\ Q(0) = 50 \end{cases}$$

(ii) The eq. is linear in Q with

$$p(t) = \frac{3}{100}, \qquad g(t) = \frac{3}{4}$$

$$\mu(t) = e^{\int \frac{3}{100} dt} = e^{\frac{3}{100} t}.$$

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]$$

$$= e^{-\frac{3}{100}t}\left[\int e^{\frac{3}{100}t} \cdot \frac{3}{4}\, dt + c\right]$$

$$= e^{-\frac{3}{100}t}\left[\frac{100}{3} e^{\frac{3}{100}t} \cdot \frac{3}{4} + c\right]$$

$$\boxed{Q(t) = 25 + c\, e^{-\frac{3}{100}t}}$$

$$50 = Q(0) = 25 + c \Rightarrow \boxed{c = 25}$$

$$\therefore \boxed{Q(t) = 25 + 25\, e^{-\frac{3}{100}t}}$$

(iii) $Q_L = \lim_{t \to \infty} Q(t) = 25.$

(iv) $Q(t) = 25.5$

$$\Rightarrow 25 + 25 e^{-\frac{3}{100}t} = 25.5$$

$$\Rightarrow 25 e^{-\frac{3}{100}t} = 0.5 \Rightarrow \boxed{t = \frac{100}{3} \ln 50}$$

<u>Ex2.</u> A tank of capacity 200 gal has initially 0.1 gm of toxic wastes dissolved in 80 gal of water. Water with toxic wastes starts flow into the tank at a rate 4 gal/min. and flow out at a rate 2 gal/min. the incoming water contains $\frac{1}{4}$ gm/gal of toxic wastes.

(a) Write the IVP that describes this process.

(b) Find the amount of toxic wastes in the tank at any time t.

(c) Find the amount of toxic wastes in the tank when it becomes to over flow.

Solution

Let $Q(t)$ be the amount

(a) of toxic wastes in the tank at any time t.

$Q(0) = 0.1$

Capicity = 200 gal.

at $t=1 \longrightarrow 82$ gal

$t=2 \longrightarrow 80 + 2(2) = 84$ gal

$t=3 \longrightarrow 80+2(3) = 86$ gal

⋮

At any time t, Volume $= 80+2t$.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dQ}{dt} = (4)\left(\frac{1}{4}\right) - (2)\left(\frac{Q(t)}{80+2t}\right)$$

So, the IVP is

(45)

$$\begin{cases} \dfrac{d\varphi}{dt} + \dfrac{1}{40+t}\varphi(t) = 1 \\ \varphi(0) = 0.1 \end{cases}$$

(b) The eq. is linear. $p(t) = \frac{1}{40+t}$, $g(t)=1$

$\cdot\ \mu(t) = e^{\int \frac{1}{40+t}dt} = e^{\ln|40+t|} = 40+t$

$$\varphi(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right]$$

$$= \frac{1}{40+t}\left[\int (40+t)\cdot 1\,dt + c\right]$$

$$= \frac{1}{40+t}\left[40t + \frac{t^2}{2} + c\right]$$

$\frac{1}{10} = \varphi(0) = \frac{1}{40}[c] \Rightarrow \boxed{c = 4}$

$$\therefore \varphi(t) = \frac{40t + \frac{t^2}{2} + 4}{40+t}$$

(c) $\varphi(60) = \dfrac{40(60) + \frac{(60)^2}{2} + 4}{40+60} = 42.04.$

Newton's Law of Cooling

States that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). that is, $\frac{du}{dt} = -k(u-T)$,

where T is the constant ambient temp.

and k is a positive constant.

$u(t)$ is the temperature of an object at any time t.

Ex. Spse that the temperature of a cup of coffee obeys Newton's Law of Cooling. If the coffee has a temperature of $90°c$ when freshly poured, and 1 min Later has cooled to $85°c$ in a room at $20°c$, determine when the coffee reaches a temperature of $65°c$.

Solution. let $u(t)$ be the temperature of the cup of coffee.

Given the IVP $\frac{du}{dt} = -k(u-T)$, where

$T = 20°C$, $u(0) = 90°C$, $u(1) = 85$

we need to find t such that $u(t) = 65°$??

So, $\frac{du}{dt} = -k(u-20) \Rightarrow \int \frac{du}{u-20} = -\int k\,dt$, $u \neq 20$

$\Rightarrow \ln|u-20| = -kt + c$

$\Rightarrow u - 20 = \pm e^{-kt+c} = \pm e^{-kt} \cdot e^{c}$

$\Rightarrow \boxed{u = 20 + Ae^{-kt}}$, where $A = \pm e^{c}$ constant.

$90 = u(0) = 20 + A \Rightarrow \boxed{A = 70}$

$85 = u(1) = 20 + 70e^{-k} \Rightarrow 70e^{-k} = 65$

$\Rightarrow e^{-k} = \frac{65}{70}$

$\Rightarrow \boxed{k = -\ln\left(\frac{65}{70}\right)}$

$\therefore u(t) = 20 + 70\, e^{\ln\left(\frac{65}{70}\right)t}$

$$\boxed{u(t) = 20 + 70\left(\frac{65}{70}\right)^{t}}$$

Now, $u(t) = 65 \Rightarrow 65 = 20 + 70\left(\frac{65}{70}\right)^t$

$$\Rightarrow \frac{45}{70} = \left(\frac{65}{70}\right)^t$$

$$\Rightarrow \ln\left(\frac{45}{70}\right) = t \ln\left(\frac{65}{70}\right)$$

$$\Rightarrow t = \ln\left(\frac{45}{70}\right) \Big/ \ln\left(\frac{65}{70}\right).$$

$$\approx 5.96 \text{ min.}$$

2.4 Difference between linear and nonlinear equations

Recall that the 1st ode has the general form

$$\boxed{\frac{dy}{dt} = f(t,y)} \quad (1)$$

If f is linear in y, then (1) is linear d.e.

If f is not linear in y, then (1) is nonlinear

Existence & Uniquence of solutions

Question. Does every IVP have exactly one solution?

Ans: For linear eqs, the answer is given by the following theorem.

Thm 2.4.1. Consider the linear d.e with the initial condition

$$\begin{cases} \frac{dy}{dt} + p(t)y = q(t) \\ y(t_0) = y_0 \end{cases} \quad (2)$$

If p and q are continuous on an open interval $I := (\alpha, \beta)$ containing $t = t_0$, then there exists aunique function $y = \phi(t)$ that satisfies the IVP (2).

Rmk: (i) thm 2.4.1 states that the given IVP has a solution and also that the problem has only one solution. In other words, the thm asserts both the existence & uniqueness of the solution of the IVP (2).

(ii) The proof of this thm is partly contained in Section 2.1 by the formula

$$y = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\, dt + c\right], \text{ where}$$

$$\mu(t) = e^{\int p(t)\, dt}.$$

Ex. Without solving, Does the following IVP have aunique solution? If so, find the largest interval in which the solution exists.

① $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases}$

Sol: Rewritting the eq. in the standard form, we have $\boxed{\frac{dy}{dt} + \frac{2}{t}y = 4t}$, so

$p(t) = \frac{2}{t}$ and $q(t) = 4t$. Notice that

p is continuous on $(-\infty, 0) \cup (0, \infty)$.

q " " " $(-\infty, \infty)$.

$\Rightarrow$ p & q are cont. on $(-\infty, 0) \cup (0, \infty)$.

the interval $(0, \infty)$ contains the initial pt ($t_0 = 1$)

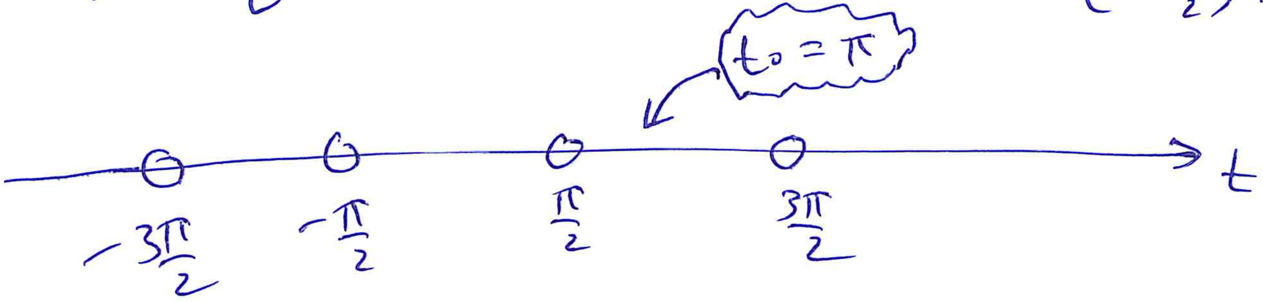
$\Rightarrow$ Thm 2.4.1 guarantees that the problem has a unique sol. on $(0, \infty)$. [the largest interval]..

② $\begin{cases} \frac{dy}{dt} + (\tan t) y = \sin t \\ y(\pi) = 0 \end{cases}$

Sol. $p(t) = \tan t$ is cont. on $(-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$

$q(t) = \sin t$ " " " $(-\infty, \infty)$.

p & q are cont. on $(-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \dots\}$



the Largest interval in which the solution is certain to exist is $(\frac{\pi}{2}, \frac{3\pi}{2})$.

H.w

(3) $\begin{cases} (\ln t)\, y' + y = \cot(t) . \\ y(2) = 3 \end{cases}$

Ans. $(1, \pi)$ [How ?!]

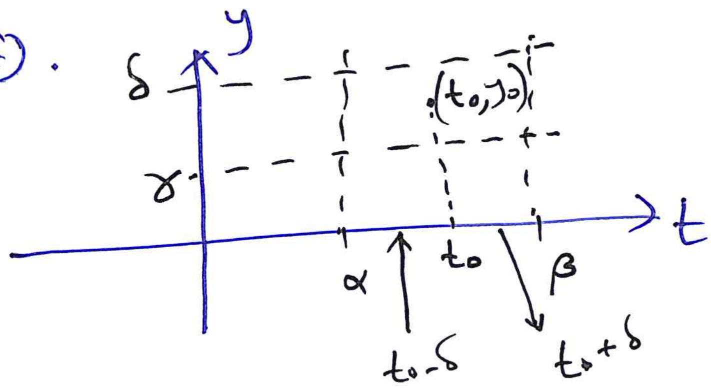
(4) $\begin{cases} y' + (\ln t)\, y = \cot(t) \\ y(2) = 3 \end{cases}$

Ans. $(0, \pi)$ How ?!

Thm 2.4.2 (Nonlinear Case)

Consider the IVP $\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ (*)

If f & $\dfrac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP (*).



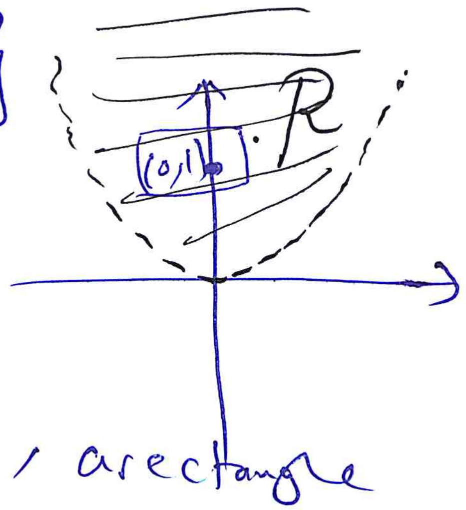
Ex. Does the IVP $\begin{cases} \dfrac{dy}{dt} = \sqrt{y - t^2} \\ y(0) = 1 \end{cases}$ have a unique solution?

Sol. $f(t,y) = \sqrt{y - t^2}$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-t^2}}$$

f & $\frac{\partial f}{\partial y}$ are continuous on the region

$$R = \{ (t,y) : y - t^2 > 0 \}$$
$$= \{ (t,y) : y > t^2 \}$$



Now $(0,1) \in R$, consequently, a rectangle can be drawn about $(0,1)$ in which f and $\frac{\partial f}{\partial y}$ are cont.

$\Rightarrow$ the IVP has a unique solution.

ex. Determine whether the thm 2.4.2 guarantees that IVP $\begin{cases} \frac{dy}{dt} = y^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$

posses a unique solution ?

Sol. $f(t,y) = y^{\frac{2}{3}}$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{y}}$$

f & $\frac{\partial f}{\partial y}$ are cont. on $R = \{(t,y) : y \neq 0\}$

the intial point $(0,0) \notin R$. Hence,

thm 2.4.2 doesnot guarantee anything.

So, in this case, we must solve the problem.

$$\frac{dy}{dt} = y^{\frac{2}{3}} \Rightarrow \int y^{-\frac{2}{3}} dy = \int dt \quad , y \neq 0$$

$$\Rightarrow 3 \cdot y^{\frac{1}{3}} = t + c$$

$$y(0) = 0 \Rightarrow 3(0) = 0 + c \Rightarrow \boxed{c = 0}$$

$$\therefore \quad 3y^{\frac{1}{3}} = t \Rightarrow \boxed{y = \left(\frac{t}{3}\right)^3 = \frac{t^3}{27}}$$

is one solution

and by inspection $y(t) = 0$ is also a solution.

$\Rightarrow$ the IVP doesnot have a unique sol.

Ex. Solve the given IVP and determine how the interval in which the solution exists depends on the initial value y_0

$$\begin{cases} \dfrac{dy}{dt} = y^2 \\ y(0) = y_0 \end{cases}$$

Sol.

$$\frac{dy}{dt} = y^2 \Rightarrow \int y^{-2}\,dy = \int dt \quad , y \neq 0$$

$$\Rightarrow -y^{-1} = t + c \Rightarrow \boxed{y = \frac{-1}{t+c}}$$

$$y(0) = y_0 \Rightarrow \frac{-1}{c} = y_0 \Rightarrow \boxed{c = \frac{-1}{y_0}}$$

$$\therefore y = \frac{-1}{t - \frac{1}{y_0}} = \frac{y_0}{1 - y_0 t}$$ is the solution of the IVP.

Now, observe that the solution becomes unbounded as $t \longrightarrow \frac{1}{y_0}$, so the interval of existence of the solution is

$$-\infty < t < \frac{1}{y_0} \quad \text{if } y_0 > 0$$

and $$\frac{1}{y_0} < t < \infty \quad \text{if } y_0 < 0$$

Bernoulli Equations

$$\boxed{\frac{dy}{dt} + p(t)y = q(t)y^n} \quad \text{—} \quad (I)$$

Notice that If $n=0$, then (I) $\Rightarrow$

$\frac{dy}{dt} + p(t)y = q(t)$ linear in y.

If $n=1$, then (I) becomes $\frac{dy}{dt} + (p(t) - q(t))y = 0$

which is seperable.

Question show that if $n \neq 0, 1$, then the substitution $\boxed{v = y^{1-n}}$ reduces (I) to a linear equation

Proof: let $v = y^{1-n}$ or $\boxed{y = v^{\frac{1}{1-n}}}$ — ①

$$\frac{dy}{dt} = \frac{1}{1-n} v^{\frac{1}{1-n}-1} \frac{dv}{dt}$$

$$\boxed{\frac{dy}{dt} = \frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dt}} \quad \text{—} \quad ②$$

Substitute ① & ② into (I),

$$\frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dt} + p(t)\, v^{\frac{1}{1-n}} = q(t)\, v^{\frac{n}{1-n}}$$

Multiply the last eq. by $(1-n)\, v^{\frac{-n}{1-n}}$,

$$\boxed{\frac{dv}{dt} + (1-n)\, p(t)\, v = (1-n)\, q(t)} \quad (II)$$

which is linear in v. ▬

Ex. Solve the d.e $t^2 y' + 2ty = y^2$, $t>0$

Sol.

$$\boxed{\frac{dy}{dt} + \frac{2}{t} y = \frac{1}{t^2} y^2} \quad (*)$$

is Bernoulli with $\boxed{n=2}$

Let $v = y^{1-n} = y^{1-2} = y^{-1}$ or $\boxed{y = v^{-1}}$ — ①

$$\boxed{\frac{dy}{dt} = -v^{-2} \frac{dv}{dt}} \quad ②$$

Substitute ① + ② into ⊛,

$$-v^{-2}\frac{dv}{dt}+\frac{2}{t}v^{-1}=\frac{1}{t^2}\left(v^{-1}\right)^2$$

Divide by $-v^{-2}$:

$$\boxed{\frac{dv}{dt}-\frac{2}{t}v=-\frac{1}{t^2}} \text{ — } (**)$$ linear in v.

$$\mu(t)=e^{-\int\frac{2}{t}dt}=e^{-2\ln|t|}=t^{-2},\ t>0.$$

$$\therefore\ v(t)=\frac{1}{t^{-2}}\left[\int t^{-2}\cdot\left(\frac{-1}{t^2}\right)dt+c\right]$$

$$\Rightarrow\ y^{-1}=t^2\left[\int -t^{-4}\,dt+c\right]$$

$$y^{-1}=t^2\left[\frac{t^{-3}}{3}+c\right]=\frac{1}{3t}+ct^2$$

$$\therefore\ \boxed{y=\frac{1}{\frac{1}{3t}+ct^2}},\ t>0.$$

H.W's Solve the following d.e's

① $(t^2+1)\frac{dy}{dt} = 4ty + 4t\sqrt{y}$.

$$\frac{dy}{dt} - \frac{4t}{t^2+1}y = \frac{4t}{t^2+1}y^{\frac{1}{2}}$$

is Bernoulli with $n=\frac{1}{2}$ .....

② $$\frac{dy}{dx} + \frac{2y}{6x+1} = -\frac{3x^2}{(6x+1)y^2}$$

is Bernoulli with $n=-2$ ...

③ $$\begin{cases} \frac{dy}{dx} = \frac{3y^2-x^2}{2xy} \\ y(1) = 2 \end{cases}$$

is Bernoulli eq. with $n=-1$

and also it is homogeneous eq.

وفقكم الله

2.6 Exact Equation and Integrating factors

Ex. Consider the DE $2x+y^2+2xy\frac{dy}{dx}=0$

This eq. is neither linear nor seperable.

Thm 2.6.1 Consider a d.e with the form

$$\boxed{M(x,y)\,dx+N(x,y)\,dy=0} \quad \text{—} \; (1)$$

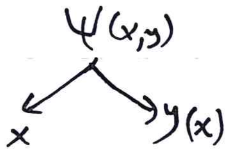
where M, N, M_y, N_x are all continuous on the Region $R: \alpha<x<\beta,\ \gamma<y<\delta$. Then Eq (1) is an exact eq. in R iff $\boxed{M_y=N_x}$. that is, there exists a function ψ satisfying $\boxed{\psi_x=M}$, $\boxed{\psi_y=N}$

iff $M_y=N_x$.

Proof. see the book.

Rmk. When $M(x,y)\,dx+N(x,y)\,dy=0$

$$\Rightarrow \psi_x\,dx+\psi_y\,dy=0$$



$$\Rightarrow \psi_x+\psi_y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{d}{dx}\psi(x,y)=0 \quad (\text{chain Rule})$$

$\Rightarrow \psi(x,y)=\text{Constant}$ defines $y=\phi(x)$ implicitly.

Ex. Back to the DE above

Solve $2x+y^2+2xy\dfrac{dy}{dx}=0$

Sol. $(2x+y^2)\,dx+2xy\,dy=0$

$M(x,y)=2x+y^2$, $N(x,y)=2xy$

$M_y=2y$, $N_x=2y \Rightarrow M_y=N_x$ exact.

by thm 2.6.1, $\exists$ a function $\psi(x,y)$ such that

$\psi_x=M(x,y)=2x+y^2$ — ①

$\psi_y=N(x,y)=2xy$ — ②

From ① $\int \psi_x(x,y)\,dx=\int(2x+y^2)\,dx$

$\Rightarrow \boxed{\psi(x,y)=x^2+y^2x+h(y)}$ — ③

$\psi_y \overset{Eq③}{=} 2yx+h'(y) \overset{Eq②}{=} 2xy$

$\Rightarrow h'(y)=0 \Rightarrow \boxed{h(y)=c_1}$ — ④

Setting ④ into ③ : $\psi(x,y)=x^2+xy^2+c_1$

Hence the solution is given implicitly

$x^2+xy^2=c$

Ex. Verify that the D.E is exact and then solve it.

$$\left(\frac{y}{1+x^2} - \frac{e^y}{x}\right) dx = \left(e^y \ln x - \tan^{-1}x + 2\right) dy.$$

Sol. $$\underbrace{\left(\frac{e^y}{x} - \frac{y}{1+x^2}\right)}_{M(x,y)} dx + \underbrace{\left(e^y \ln x - \tan^{-1}x + 2\right)}_{N(x,y)} dy = 0$$

$$M_y = \frac{e^y}{x} - \frac{1}{1+x^2} \quad , N_x = e^y \cdot \frac{1}{x} - \frac{1}{1+x^2}$$

$$\Rightarrow M_y = N_x \quad \text{exact.}$$

Thus, $\exists$ a function $\psi(x,y)$ s.t

$$\psi_x = M = \frac{e^y}{x} - \frac{y}{1+x^2} \quad \text{—— (1)}$$

$$\psi_y = N = e^y \ln x - \tan^{-1}x + 2 \quad \text{—— (2)}$$

From (2): $$\int \psi_y \, dy = \int \left(e^y \ln x - \tan^{-1}x + 2\right) dy$$

$$\boxed{\psi(x,y) = e^y \ln x - y\tan^{-1}x + 2y + h(x)} \quad (3)$$

$$\Rightarrow \psi_x \overset{(3)}{=} e^y \cdot \cancel{\frac{1}{x}} - y \cdot \cancel{\frac{1}{1+x^2}} + h'(x)$$

$$\overset{(2)}{=} \cancel{\frac{e^y}{x}} - \cancel{\frac{y}{1+x^2}}$$

$$\Rightarrow h'(x) = 0 \Rightarrow \boxed{h(x) = c} \text{ — } (4)$$

$\therefore$ (4) in (3) $\Rightarrow$ the solution is given implicitly $e^y \ln x - y \tan^{-1} x + 2y = c$

H.W's (1) Solve the d.e

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0$$

Ans. $y \sin x + x^2 e^y - y = c.$

Integrating Factors

Nonexact made exact

Consider the D.E $\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$ $(*)$

Spse that $(*)$ is not exact ($M_y \neq N_x$).

It is sometimes possible to make the d.e $(*)$ exact. Multiply both sides of $(*)$ by appropriate integrating factor $\mu(x,y)$:

$$\boxed{\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0} \quad (**)$$

Eq $(**)$ is exact iff $(\mu M)_y = (\mu N)_x$

$$\iff \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\iff \boxed{\mu_y M - \mu_x N + \mu(M_y - N_x) = 0} \quad (***)$$

$(***)$ is a 1st order pde

① If $\dfrac{M_y - N_x}{N} = f(x)$ "function of x alone"

then $\mu(x) = e^{\int f(x)\,dx}$

② If $\frac{M_y - N_x}{M} = g(y)$ " function of y alone"

then $\mu(y) = e^{-\int g(y)\,dy}$

③ If $\frac{N_x - M_y}{xM - yN} = h(xy)$, then

$$\mu(xy) = e^{\int h(xy)\,d(xy)}$$

i.e $\mu(u) = e^{\int h(u)\,du}$, where $u = xy$.

Ex. show that the d.e

$$(3x^2y - 8x)\,y' = 4y - 2xy^2$$ is not exact.

Then find an appropriate integrating factor which can be used to make it exact. Find the new exact eq. & solve it.

sol. $\underbrace{(3x^2y - 8x)}_{N}\,dy + \underbrace{(2xy^2 - 4y)}_{M}\,dx = 0$ —— ①

$$M_y = 4xy - 4, \quad N_x = 6xy - 8$$

- $\dfrac{M_y - N_x}{N} = \dfrac{(4xy-4) - (6xy-8)}{3x^2y - 8x}$

$= \dfrac{4 - 2xy}{3x^2y - 8x}$ $\neq$ function of x alone & $\neq$ " " y alone

- $\dfrac{M_y - N_x}{M} = \dfrac{(4xy-4) - (6xy-8)}{2xy^2 - 4y}$

$= \dfrac{4 - 2xy}{2y(xy-2)} = \dfrac{-2(xy-2)}{2y(xy-2)}$

$= -\dfrac{1}{y}.$

$\therefore \mu(y) = e^{-\int -\frac{1}{y}\,dy} = e^{\ln|y|} = y, \; y > 0.$

Multiply both sides of ① by $\mu(y) = y$:

$$\boxed{(3x^2y^2 - 8xy)\,dy + (2xy^3 - 4y^2)\,dx = 0} \quad ②$$

the new exact eq.

$\exists$ a function $\psi(x,y)$ s.t.

$$\psi_x = 2xy^3 - 4y^2 \quad \text{---} \; (B)$$

$$\psi_y = 3x^2y^2 - 8xy \quad \text{---} \; (4)$$

From (4): $\int \psi_y \, dy = \int (3x^2y^2 - 8xy) \, dy$

$$\psi(x,y) = x^2y^3 - 4xy^2 + h(x) \quad \text{---} \; (5)$$

$$\psi_x \overset{(5)}{=} 2xy^3 - 4y^2 + h'(x) \overset{(4)}{=} 2xy^3 - 4y^2$$

$$\Rightarrow h'(x) = 0 \Rightarrow \boxed{h(x) = c}.$$

$\therefore$ the solution is $\boxed{x^2y^3 - 4xy^2 = c}$

ex. Solve $\boxed{(x+2)\sin y \, dx + (x \cos y) \, dy = 0}$ (1)

sol. $M = (x+2)\sin y \Rightarrow M_y = (x+2)\cos y$.

$N = x \cos y \Rightarrow N_x = \cos y$.

$\Rightarrow M_y \neq N_x$ not exact.

- $$\frac{M_y - N_x}{M} = \frac{(x+2)\cos y - \cos y}{(x+2)\sin y}$$

$$= \frac{(x+2-1)\cos y}{(x+2)} = \frac{(x+1)\cos y}{x+2}$$

$$\frac{M_y - N_x}{N} = \frac{(x+1)\cos y}{x\cos y} = 1 + \frac{1}{x}$$

$$\therefore \mu(x) = e^{\int (1+\frac{1}{x})\,dx} = e^{x + \ln|x|}$$
$$= e^x \cdot e^{\ln|x|}$$
$$= x e^x, \quad x > 0.$$

Multiply both sides of ① by $\mu(x) = x e^x$

$\boxed{x(x+2)e^x \sin y\, dx + x^2 e^x \cos y\, dy = 0}$ is the new exact eq.

$\exists\ \psi(x,y)$ s.t $\psi_x = x(x+2)e^x \sin y$ — ②

$\psi_y = x^2 e^x \cos y$ — ③

From ③: $\int \psi_y\, dy = \int (x^2 e^x \cos y)\, dy$

$$\boxed{\psi(x,y) = x^2 e^x \sin y + h(x)} \quad ④$$

$$\psi_x \overset{④}{=} \cancel{2xe^x \sin y} + \cancel{x^2 e^x \sin y} + h'(x)$$
$$\overset{②}{=} \cancel{x^2 e^x \sin y} + \cancel{2xe^x \sin y}$$

$\Rightarrow h'(x) = 0 \Rightarrow \boxed{h(x) = c}$

$\Rightarrow$ the solution is given implicitly by $\boxed{x^2 e^x \sin y = c}$

H.W's Solve the following IVPs.

① $\begin{cases} xy^3 + (x^2y^2+1)\,y' = 0 \\ y(2) = 1\ ,\ x>0\ ,\ y>0 \end{cases}$ (Nonexact made exact)

and Bernoulli with $n=-1$.

Ans. $\frac{1}{2}x^2y^2 + \ln y = 2$

② $\begin{cases} y' = \dfrac{3y^2-x^2}{2xy} \\ y(1) = 2 \end{cases}$. Write y as a function of x.

It is nonexact made exact.

Homogeneous and Bernoulli with $n=-1$

Ans. $y = x\sqrt{3x+1}$

③ $\begin{cases} 2x^2+y + (x^2y - x)\,y' = 0 \\ y(1) = 1 \end{cases}$

2.8 (The existence & Uniquence theorm)
2.9 (Some special second order d.e's)

Thm 2.8.1 Consider the IVP

$$\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(0) = 0 \end{cases} \quad (1)$$

If f & $\frac{\partial f}{\partial y}$ are continuous in a rectangle

$R: |t| \le a,\ |y| \le b$, then there is some

interval $|t| \le h \le a$ in which there

exists a unique solution $y = \phi(t)$ of

the IVP (1).

Method of Successive approximations or Picards iteration method

To use this method, we generate a sequence of functions $\{\phi_n(t)\}$ where $\phi_n(t)$ satisfy the following integral equation

$$\boxed{y = \phi(t) = \int_0^t f(s, \phi(s))\, ds} \quad (2)$$

Note that eq② is exactly the same as eq①. We will use this method by choosing an initial function $\phi_0(t)$. The simplest choice is $\boxed{\phi_0(t) = 0}$.

Next, $\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$\vdots$$

$$\phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\, ds.$$

If $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ conv., then

$y = \phi(t)$ will be the solution of the IVP①.

Ex. Solve the IVP $\begin{cases} y' = 2t(1+y) \\ y(0) = 0 \end{cases}$ by the method of successive approximations. (or Picard's method).

Sol: $f(t,y) = 2t(1+y)$. Choose $\boxed{\phi_0(t) = 0}$

Next $\Phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$

$$= \int_0^t f(s, 0)\, ds = \int_0^t 2s\, ds = s^2 \Big|_0^t = t^2$$

$\therefore$ $\boxed{\Phi_1(t) = t^2}$

$$\Phi_2(t) = \int_0^t f(s, \Phi_1(s))\, ds$$

$$= \int_0^t f(s, s^2)\, ds = \int_0^t 2s(1+s^2)\, ds$$

$$= s^2 + \frac{2s^4}{4} \Big|_0^t$$

$$\Rightarrow \boxed{\Phi_2(t) = t^2 + \frac{1}{2}t^4}$$

$$\Phi_3(t) = \int_0^t f(s, \phi_2(s))\, ds =$$

$$= \int_0^t f\left(s, s^2 + \frac{1}{2}s^4\right) ds$$

$$= \int_0^t 2s\left(1 + s^2 + \frac{1}{2}s^4\right) ds$$

$$= s^2 + \frac{1}{2}s^4 + \frac{s^6}{6} \Big|_0^t$$

$$\boxed{\phi_3(t) = t^2 + \frac{1}{2}t^4 + \frac{t^6}{6}}$$

:

$$\Phi_n(t) = t^2 + \tfrac{1}{2}t^4 + \tfrac{1}{6}t^6 + \cdots + \frac{t^{2n}}{n!}$$

$$= t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots + \frac{(t^2)^n}{n!}$$

$$\boxed{\Phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}} \quad \text{—} \ (*)$$

It follows from $(*)$ that $\Phi_n(t)$ is the n^{th} partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{k!} \quad \cdots \ (**)$$

Hence $\lim_{n\to\infty} \Phi_n(t)$ exists iff the series $(**)$ converges. Applying the ratio test, we see

$$\lim_{k\to\infty} \left| \frac{t^{2k+2}}{(k+1)!} \, \frac{k!}{t^{2k}} \right|$$

$$= t^2 \lim_{k\to\infty} \frac{1}{k+1} = t^2 \cdot 0 < 1$$

for all t.

thus the series (~~**~~) Converges for all t

and $\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t^{2k}}{k!}$

$$= \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$$

$$= e^{t^2} - 1$$

$\therefore$ $y = \phi(t) = e^{t^2} - 1$ is the solution of the IVP.

Rmk. We use $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$$\Rightarrow x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x - 1$$

i.e, $\sum_{k=1}^{\infty} \frac{x^k}{k!} = e^x - 1$

H.W ① Use Picard's method to solve the IVP $\begin{cases} y' = 3y + 3 \\ y(0) = 0 \end{cases}$

Ans. $y = e^{3t} - 1$.

(2) Same for the IVP $\begin{cases} y' = y+1-t \\ y(0)=0 \end{cases}$.

<u>Ex.</u> Transform the following IVP

$$\begin{cases} \dfrac{dy}{dt} = 2t^2 + y^2 \\ y(1) = 2 \end{cases}$$ into an equivalent

problem with the initial point at the orgin.

<u>Sol.</u> Let $w(s) = y(t) - 2$, $s = t-1$

that is $\boxed{w(t-1) = y(t) - 2}$

when $t=1$, $w(0) = y(1) - 2 = 2-2 = 0$

$$\Rightarrow \boxed{w(0) = 0}$$

Now, $\boxed{y = w(s) + 2}$, $s = t-1$

$$\frac{dy}{dt} = \frac{dw}{ds} \cdot \frac{ds}{dt} = \frac{dw}{ds}$$

$\therefore$ $\dfrac{dy}{dt} = 2t^2 + y^2 \Rightarrow \dfrac{dw}{ds} = 2(s+1) + (w(s)+2)^2$

$\therefore$ the IVP becomes $\begin{cases} \dfrac{dw}{ds} = 2(s+1) + (w+2)^2 \\ w(0) = 0 \end{cases}$

2.9 (Exercises 36 — 51)

Some special Second order Eqs

The general form of the 2^{nd} order d.e is $\boxed{y'' = f(t,y,y')}$ — (1)

There are two types of eq (1). that can be tranformed into 1^{st} order eqs by suitable change of variable.

Case 1. Equations with the dependent variable missing. $\boxed{y'' = f(t,y')}$ (2)

Let $y' = v$, then $y'' = v'$

Eq (2) becomes $v' = f(t,v)$ which is 1^{st} order d.e.

Ex. (36) Solve $t^2 y'' + 2ty' = 1$, $t > 0$.

let $v = y'$, $v' = y''$

$\Rightarrow t^2 v' + 2tv = 1$

$$\Rightarrow \frac{dv}{dt} + \frac{2}{t}v = t^{-2}, \; t>0$$ linear in v.

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = t^2, \; t>0$$

$$\therefore v = \frac{1}{\mu(t)}\left[\int q(t)\mu(t)\,dt + c\right]$$

$$= \frac{1}{t^2}\left[\int t^{-2}.t^2\,dt + c\right] = \frac{1}{t^2}[t+c]$$

$$\Rightarrow v = \frac{1}{t} + \frac{c}{t^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{t} + ct^{-2}$$

$$\Rightarrow \int dy = \int\left(\frac{1}{t} + ct^{-2}\right)dt$$

$$\Rightarrow \boxed{y = \ln t + k_1 t^{-1} + k_2 \quad t>0}$$

ex. (51) Solve $\begin{cases} y'y'' - t = 0 \\ y(1) = 2, \; y'(1) = 1 \end{cases}$

Sol. Let $y' = v \Rightarrow y'' = v'$

$$\Rightarrow vv' = t \Rightarrow \int v\,dv = \int t\,dt$$

$$\Rightarrow \boxed{\frac{V^2}{2} = \frac{t^2}{2} + C_1}$$

$V(1) = y'(1) = 1$ gives

$$\frac{(1)^2}{2} = \frac{(1)^2}{2} + C_1 \Rightarrow \boxed{C_1 = 0}$$

$$\therefore \frac{V^2}{2} = \frac{t^2}{2} \Rightarrow V = t$$

$$\Rightarrow \int dy = \int t \, dt$$

$$\Rightarrow \boxed{y = \frac{t^2}{2} + C_2}$$

$$y(1) = 2 \Rightarrow 2 = \frac{(1)^2}{2} + C_2 \Rightarrow \boxed{C_2 = 3/2}$$

$$\therefore \boxed{y = \frac{t^2}{2} + \frac{3}{2}}$$

Case 2. Missing t in (1), i.e.,

$$y'' = f(y, y')$$

Let $y' = v$, $y'' = v'$

Ex 42 Solve $y y'' + (y')^2 = 0$

Let $y' = v \Rightarrow y'' = v'$

Substitute: $y v' + v^2 = 0$

or $y\dfrac{dv}{dt}+v^2=0$.

$y\left(\dfrac{dv}{dy}\dfrac{dy}{dt}\right)+v^2=0$ (Chain Rule)

$y\dfrac{dv}{dy}\cdot v+v^2=0$ (seperable)

$\Rightarrow \boxed{\dfrac{1}{v}\,dv=-\dfrac{1}{y}\,dy \,,\ v\neq 0}$

$\Rightarrow \ln|v|=-\ln|y|+c$

$\Rightarrow v=Ay^{-1}$

i.e $v=\dfrac{A}{y}$

$\Rightarrow \dfrac{dy}{dt}=\dfrac{A}{y}$

$\Rightarrow \int y\,dy=A\int dt$

$\boxed{\dfrac{y^2}{2}=At+c}$

If $\boxed{v=0}\Rightarrow y'=0\Rightarrow \boxed{y=k}$ " Constant" Satisfy the eq.

H.W Solve the IVP $\begin{cases} yy''=(y')^2-(y')^3 \\ y(0)=1,\ y'(0)=2 \end{cases}$

CH3 Second order linear Equations

3.1 Homogeneous Equations with Constant Coefficients

3.3 Complex roots of the characterstic eqs

A second order ode has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad — ①$$

· Eq ① is said to be linear if f has the form $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y$ — ②

(i.e., if f is linear in y and y')

So, eq ① can be rewritten as

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \quad — ③$$

· If eq ① is not of the form ③, then it is called nonlinear

If $g(t) = 0$ in ③, then it is called homogeneous

If $g(t) \neq 0$ in ③, " " " " nonhomog.

In sections 3.1, 3.3, 3.4 (part), we seek the solution of the following 2nd order lin. homog eq. with constant coefficients

$$ay'' + by' + cy = 0, \quad \text{— } (4)$$

where $a, b, \& c$ are constants.

To solve (4), we assume the solution as

$$y = e^{rt} \Rightarrow y' = re^{rt}, \quad y'' = r^2 e^{rt}$$

Substitute into (4):

$$(ar^2 + br + c)\, e^{rt} = 0$$

$\Rightarrow ar^2 + br + c = 0$ is called the characteristic or auxilliary eq.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

. So, we have three cases

Case 1 r_1, r_2 are distinct real roots.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Case 2 $\quad r_1 = r_2 = r$ (repeated real roots)

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

Case 3. $\quad r_1, r_2$ are Conjugate complex roots.

$$r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i.$$

$$y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

Ex. Solve the following d.e's.

① $y'' + 3y' + 2y = 0.$

The aux. eq. is $r^2 + 3r + 2 = 0$

$\Rightarrow (r+2)(r+1) = 0 \Rightarrow r_1 = -2, \; r_2 = -1.$

$$y = c_1 e^{-2t} + c_2 e^{-t}.$$

② $y'' + 5y' + 6y = 0 \; ; \; y(0) = 2 = y'(0).$

The characterstic eq. is $r^2 + 5r + 6 = 0$

$\Rightarrow (r+3)(r+2) = 0 \Rightarrow r_1 = -3, \; r_2 = -2$

$$\boxed{y = c_1 e^{-3t} + c_2 e^{-2t}}$$

$$2 = y(0) = c_1 + c_2 \Rightarrow \boxed{c_1 + c_2 = 2} \text{ --- (A)}$$

$$y' = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

$$2 = y'(0) = -3c_1 - 2c_2 \Rightarrow \boxed{-3c_1 - 2c_2 = 2} \text{ --- (B)}$$

$\underline{2\,Eq(A) + Eq(B):}$ $\quad -c_1 = 6 \Rightarrow \boxed{c_1 = -6}$

$$\Rightarrow \boxed{c_2 = 8}$$

$$\therefore y = -6e^{-3t} + 8e^{-2t}.$$

(3) $\quad y'' + 6y' + 9y = 0$

The aux. eq. $r^2 + 6r + 9 = 0$

$$\Rightarrow (r+3)^2 = 0 \Rightarrow r = -3, -3$$

$$y = c_1 e^{-3t} + c_2 t e^{-3t}.$$

(4) $\quad y'' + y' + 9.25y = 0, \quad y(0) = 0, \quad y'(0) = 9.$

The aux. eq. is $r^2 + r + 9.25 = 0$

$$r = \frac{-1 \pm \sqrt{1-4(1)(9.25)}}{2}$$

$$= \frac{-1 \pm \sqrt{-36}}{2} = \frac{-1}{2} \pm \frac{6i}{2}$$

$$= -\frac{1}{2} \pm 3i .$$

$$y = c_1 e^{-\frac{1}{2}t} \cos 3t + c_2 e^{-\frac{1}{2}t} \sin 3t .$$

$$0 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow \boxed{c_1 = 0}$$

$$y = c_2 e^{-\frac{1}{2}t} \sin 3t .$$

$$y' = -\frac{1}{2} c_2 e^{-\frac{1}{2}t} \sin 3t + 3c_2 e^{-\frac{1}{2}t} \cos 3t .$$

$$9 = y'(0) = 0 + 3c_2 \cdot 1 \cdot 1 \Rightarrow \boxed{c_2 = 3}$$

$$\therefore \boxed{y = 3e^{-\frac{1}{2}t} \sin 3t}$$

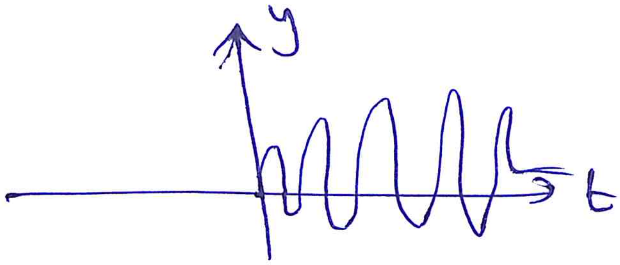
. Discuss the behavior.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 3e^{-\frac{1}{2}t} \sin 3t = 0,$$

since $\underbrace{-3e^{-\frac{1}{2}t}}_{\xrightarrow[t\to\infty]{} 0} \le 3e^{-\frac{1}{2}t} \sin 3t \le \underbrace{3e^{-\frac{1}{2}t}}_{\xrightarrow[t\to\infty]{} 0}$

$\therefore \lim_{t\to\infty} 3e^{-\frac{1}{2}t}\sin 3t = 0$, by sequeeze thm.

this is called decay oscillation



⑥ Solve $\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = 0, \ y'(0) = 1 \end{cases}$

Sol. the aux. eq. is $16r^2 - 8r + 145 = 0$

$$\Rightarrow r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{2(16)}$$

$$= \frac{8 \pm \sqrt{64(1-145)}}{32}$$

$$= \frac{8 \pm 8(12)i}{32} = \frac{1}{4} \pm 3i$$

$$y = c_1 e^{\frac{1}{4}t}\cos 3t + c_2 e^{\frac{1}{4}t}\sin 3t.$$

$0 = y(0) = c_1 \cdot 1 \cdot 1 + c_2 \cdot 0 \Rightarrow \boxed{c_1 = 0}$

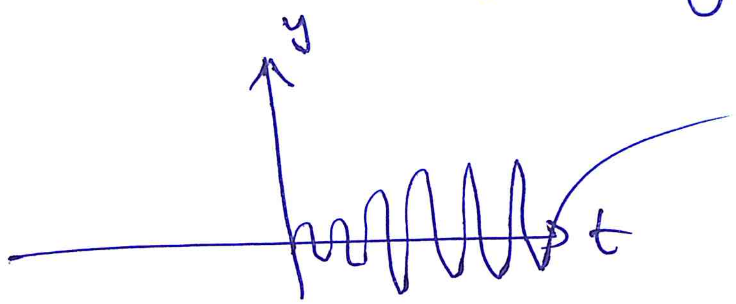
$$y = c_2 e^{\frac{1}{4}t} \sin 3t$$

$$y' = c_2\left[\frac{1}{4} e^{\frac{1}{4}t} \sin 3t + 3 e^{\frac{1}{4}t} \cos 3t \right]$$

$$1 = y'(0) = c_2 [\, 0 + 3 \,] \Rightarrow \boxed{c_2 = \frac{1}{3}}$$

$$\therefore \boxed{y = \frac{1}{3} e^{\frac{1}{4}t} \sin 3t}$$

$\lim_{t \to \infty} y(t) =$ unbounded this is called growing oscillation



ex. let y be the solution of the IVP

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha, \; y'(0) = 1. \end{cases}$$

Find α for which $\lim_{t \to \infty} y(t) = 0$.

Sol. the aux. eq. is $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, \; r_2 = -1$$

$$y = c_1 e^{2t} + c_2 e^{-t}$$

$$\alpha = y(0) = c_1 + c_2 \quad \text{— } (1)$$

$$y' = 2c_1 e^{2t} - c_2 e^{-t}$$

$$1 = y'(0) = 2c_1 - c_2 \quad \text{— } (2)$$

eq (1) + eq (2): $\alpha + 1 = 3c_1 \Rightarrow \boxed{c_1 = \frac{\alpha+1}{3}}$

$$\therefore c_2 = \alpha - \frac{\alpha+1}{3} = \frac{3\alpha - \alpha - 1}{3} = \frac{2\alpha - 1}{3}$$

$$\therefore y = \left(\frac{\alpha+1}{3}\right) e^{2t} + \left(\frac{2\alpha-1}{3}\right) e^{-t}$$

since $\lim_{t\to\infty} y(t) = 0$ and $e^{2t} \xrightarrow[t\to\infty]{} \infty$ as,

then $\frac{\alpha+1}{3}$ must be 0

i.e., $\frac{\alpha+1}{3} = 0 \Rightarrow \boxed{\alpha = -1}$

H.W's

(1) $\begin{cases} y'' + y' - 2y = 0 \\ y(0) = \beta,\ y'(0) = 2. \end{cases}$

Find β for which $\lim_{t \to \infty} y(t) = 0$.

(2) Consider $y'' + 2\alpha y' + y = 0$.
Assume that the aux. eq. has complex roots. Find α for which $\lim_{t \to \infty} y(t) = 0$.

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

ex. Use the Euler's formula to write the given expression in the form $a+bi$.

(1) $e^{3i\pi} = e^{i(3\pi)} = \cos 3\pi + i\sin 3\pi$
$= -1 + i(0) = -1.$

(2) $e^{2+\frac{\pi}{2}i} = e^2 \cdot e^{\frac{\pi}{2}i} = e^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
$= e^2(0 + i) = e^2 i$

(3) π^{-1+2i}

$$= e^{(-1+2i)\ln\pi}$$

$$= e^{-\ln\pi}\, e^{(2\ln\pi)i}$$

$$= \frac{1}{\pi}\left[\cos(2\ln\pi) + i\sin(2\ln\pi)\right]$$

ex. Use Euler's formula, to show that

(a) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, (b) $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Pf. (b) R.H.S $= \dfrac{e^{ix} - e^{-ix}}{2i}$

$$= \frac{(\cos x + i\sin x) - (\cos(-x) + i\sin(-x))}{2i}$$

$$= \frac{\cancel{\cos x} + i\sin x - \cancel{\cos x} + i\sin x}{2i}$$

$$= \frac{2i\sin x}{2i} = \sin x = \text{L.H.S}.$$

(a) Exercise.

Euler Equations (Exercises in Sec. 3.3)

The general form of homog. Euler Eq. is

$$At^2 y'' + Bty' + Cy = 0, \quad t > 0 \quad \text{—} (*)$$

where $A, B, C \in \mathbb{R}$ are constants.

Let $x = \ln t$ or $t = e^x$.

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}} \quad \text{—} (1)$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right)$$

$$= \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt}$$

$$= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\cdot\frac{1}{t}$$

$$\boxed{\frac{d^2y}{dt^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)} \quad \text{—} (2)$$

Substitute (1) & (2) into $(*)$:

$$A\cancel{t^2}\cdot\frac{1}{\cancel{t^2}}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + B\cancel{t}\cdot\frac{1}{\cancel{t}}\frac{dy}{dx} + cy = 0$$

$$\Rightarrow \boxed{A\frac{d^2y}{dx^2} + (B-A)\frac{dy}{dx} + Cy = 0} \quad (**)$$

Notice that $(**)$ is homog. 2nd order with constant coefficients.

Ex: Solve $t^2y'' + ty' + y = 0$.

let $\boxed{x = \ln t}$ and using $(**)$ with $A=1, B=1, C=1$.

$$\frac{d^2y}{dx^2} + y = 0$$

The aux. eq. is $r^2+1=0 \Rightarrow r=\pm i$

$$y = c_1 e^{0x}\cos x + c_2 e^{0x}\sin x$$

$$= c_1 \cos(\ln t) + c_2 \sin(\ln t), \quad t>0.$$

ex. Solve $4t^2y'' + 12ty' + 5y = 0$.

~~the~~ Let $x = \ln t$, then the eq. becomes

$$4\frac{d^2y}{dx^2} + (12-4)\frac{dy}{dx} + 5y = 0$$

$$\boxed{4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 0.}$$

the aux. eq. is $4r^2+8r+5=0$

$$r = \frac{-8 \pm \sqrt{64-4(4)(5)}}{2(4)}$$

$$= \frac{-8 \pm 4i}{8} = -1 \pm \tfrac{1}{2}i$$

$$y = c_1 e^{-x}\cos\left(\tfrac{1}{2}x\right) + c_2 e^{-x}\sin\left(\tfrac{1}{2}x\right)$$

$$= c_1 e^{-\ln t}\cos\left(\tfrac{1}{2}\ln t\right) + c_2 e^{-\ln t}\sin\left(\tfrac{1}{2}\ln t\right)$$

$$= \frac{c_1}{t}\cos\left(\tfrac{1}{2}\ln t\right) + \frac{c_2}{t}\sin\left(\tfrac{1}{2}\ln t\right), \quad t>0.$$

3.2 Solutions of linear homogeneous Equations, the Wronskian.

Theorem 3.2.1 (Existence and uniqueness)

Consider the IVP $\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$ $\boxed{I}$

If p, q, and g are continuous functions on an open interval $I = (\alpha, \beta)$ containing t_0, then IVP $\boxed{I}$ has exactly one solution.

Ex. Find the largest interval in which the solution of the IVP $\begin{cases} (t^2-3t)y'' + ty' - (t+3)y = 0 \\ y(1) = 2, \quad y'(1) = 1 \end{cases}$ is certain to exist.

Sol.
$$y'' + \frac{t}{t(t-3)}y' - \frac{(t+3)}{t(t-3)}y = 0$$

$$p(t) = \frac{t}{t(t-3)}, \quad q(t) = \frac{-(t+3)}{t(t-3)}, \quad g(t) = 0$$

p, q, and g are continuous on $(-\infty, 0) \cup (0,3) \cup (3, \infty)$

the largest interval containing $t_0 = 1$ is $(0,3)$ in which the solution is certain to exist.

Thm 3.2.2 (Principle of Superposition)

If y_1 and y_2 are two solutions of the d.e $L[y] = y'' + p(t)y' + q(t)y = 0$, then the linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution of the d.e $L[y] = 0$, for any values of c_1 and c_2, where $L[y]$ is a differential operator using for simplicity.

proof: We need to prove that if $L[y_1] = 0$, and $L[y_2] = 0$, then $L[c_1 y_1 + c_2 y_2] = 0$, for any values of c_1 & c_2. Indeed,

$$
\begin{aligned}
L[c_1 y_1 + c_2 y_2] &= (c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' \\
&\quad + q(t)(c_1 y_1 + c_2 y_2) \\
&= c_1 y_1'' + c_2 y_2'' + c_1 p(t) y_1' + c_2 p(t) y_2' \\
&\quad + c_1 q(t) y_1 + c_2 q(t) y_2 \\
&= c_1 [y_1'' + p(t) y_1' + q(t) y_1] \\
&\quad + c_2 [y_2'' + p(t) y_2' + q(t) y_2] \\
&= c_1 L[y_1] + c_2 L[y_2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \quad (\text{since } y_1 \,\&\, y_2 \text{ are solutions}) \\
&= 0
\end{aligned}
$$

Df: the <u>Wronskian</u> of the solutions y_1 and y_2 is given by $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

ex. Find $W(t, t\ln t)$, $t>0$.

Sol. $y_1 = t \Rightarrow y_1' = 1$, $y_2 = t\ln t \Rightarrow y_2' = t\cdot\frac{1}{t} + \ln t\cdot 1$
$= 1 + \ln t$.

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t\ln t \\ 1 & 1+\ln t \end{vmatrix}$$

$$= t(1+\ln t) - (t\ln t)(1)$$
$$= t + t\ln t - t\ln t = t.$$

<u>Thm 3.2.3</u> Spse that y_1 & y_2 are solutions for the IVP

$$\begin{cases} L[y] = y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases} \quad (*)$$

by thm 3.2.2, $\boxed{y = c_1 y_1 + c_2 y_2}$ $(**)$ is also a solution. To find c_1, c_2, we use the initial conditions $(**)$. Then we have the following: the solution $(**)$ satisfies $(*)$ if and only if $W(t_0) \neq 0$.

Thm 3.2.4 Spse that y_1 and y_2 are two solutions of the d.e $L[y] = y'' + p(t)y' + q(t)y = 0$ then the family of solutions $y = c_1 y_1 + c_2 y_2$ with c_1, c_2 arbitrary includes every solution of Eq. $L[y] = 0$ iff there exists a point t_0 where $W(y_1, y_2) \neq 0$.

Df. We say that y_1 & y_2 are linearly independent on I iff $W(y_1, y_2)(t) \neq 0$, for at least one $t \in I$.

Ex. Are $\{y_1, y_2\}$ lin. indep? where $y_1 = e^{2t}$, $y_2 = e^{3t}$.

Ans. $W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 3e^{5t} - 2e^{5t}$
$= e^{5t} \neq 0, \forall t \in \mathbb{R}$

$\therefore \{e^{2t}, e^{3t}\}$ are lin. indep. on $(-\infty, \infty)$.

ex. Are $\{1, x, x^2\}$ lin. indep.?

$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2)$
$= 2 \neq 0$

$\therefore \{1, x, x^2\}$ are lin. indep. on $(-\infty, \infty)$.

Rmk: If $\{y_1, y_2\}$ are lin. indep., then $W(y_1,y_2)=0$ but the converse is not true.

(H.w) Give a counterexample.

Df: (Fundamental set of solutions)

the solutions y_1 and y_2 are said to form a fundamental set of solutions of the eq.

$L[y] = y'' + p(t)y' + q(t)y = 0$ iff $W(y_1, y_2) \neq 0$

Ex. verify that $y_1 = t^2$, $y_2 = t^{-1}$ form a fundamental set of solutions for the d.e $\boxed{t^2y'' - 2y = 0,\ t>0}$

sol: • Verification $y_1 = t^2$, $y_1' = 2t$, $y_1'' = 2$

$\therefore t^2 y_1'' - 2y_1 = t^2(2) - 2(t^2) = 0$ ✓

$y_2 = t^{-1}$, $y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$

$\therefore t^2 y_2'' - 2y_2 = t^2(2t^{-3}) - 2t^{-1} = 2t^{-1} - 2t^{-1} = 0$ ✓

$\therefore y_1 = t^2$, $y_2 = t^{-1}$ are solutions.

• $W(y_1,y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -t^2 \cdot t^{-2} - (2t)(t^{-1})$

$= -1 - 2 = -3 \neq 0,\ \forall t$

$\Rightarrow \{t^2, t^{-1}\}$ form a fundamental set of solutions.

H-W's ① Check if $y_1 = x, y_2 = xe^x$ form a fundamental set of solutions for the d.e

$$x^2 y'' - x(x+2)y' + (x+2)y = 0 , \quad x > 0 .$$

② check if $y_1 = x$, $y_2 = \sin x$ form a fundamental set of solutions for the d.e

$$(1 - x\cot x)y'' - xy' + y = 0 , \quad 0 < x < \pi .$$

Thm 3.2.5 Consider the d.e $L[y] = y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on some open interval I. Choose some point $t_0 \in I$. Let y_1 be the solution of $L[y] = 0$ and satisfies $y_1(t_0) = 1$, $y_1'(t_0) = 0$, and let y_2 be the solution of $L[y] = 0$ that satisfies $y_2(t_0) = 0$, $y_2'(t_0) = 1$. Then y_1 and y_2 form a fundamental set of solutions of $L[y] = 0$.

Rmk: (on thm 3.2.5), $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$

Hence by this thm, we need to observe that the existence of the functions y_1 and y_2.

Ex: Find the fundamental set of solutions specified by thm 3.2.5 for $y''-y=0$ using $t_0=0$.

Sol: The aux. eq. is $r^2-1=0 \Rightarrow r=\pm 1$

$\boxed{y_c = c_1 e^t + c_2 e^{-t}}$ let $y_1 = e^t$, $y_2 = e^{-t}$

let $y_1(0)=1$, $y_1'(0)=0$. this means

$c_1+c_2=1$, $c_1-c_2=0 \Rightarrow 2c_1=1 \Rightarrow c_1=\frac{1}{2}$
$c_2=\frac{1}{2}$

$\therefore y_3 = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \frac{e^t+e^{-t}}{2} = \cosh t.$

Also, let $y_2(0)=0$, $y_2'(0)=1$

$\Rightarrow c_1+c_2=0$, $c_1-c_2=1 \Rightarrow c_1=\frac{1}{2}$, $c_2=\frac{-1}{2}$

$\therefore y_4 = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t$

$\therefore \{y_3, y_4\} = \{\cosh t, \sinh t\}$ form a fundamental set of solutions.

Thm 3.2.6 (Abel's thm)

If y_1 and y_2 are solutions of the d-e

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$ where

p, q are continuous on some open interval I,

then $W(y_1, y_2) = c\, e^{-\int p(t)\,dt}$, where c

is constant that depends on y_1 and y_2

but not on t. Moreover, If $c = 0$, then

$W(y_1, y_2) = 0, \ \forall t \in I$.

If $c \neq 0$, $W(y_1, y_2) \neq 0, \ \forall t \in I$.

Proof. Since y_1 and y_2 are solution of $L[y] = 0$,

then $y_1'' + p(t)y_1' + q(t)y_1 = 0$ — (1)

$y_2'' + p(t)y_2' + q(t)y_2 = 0$ — (2)

Multiply Eq (1) by $-y_2$ and Eq (2) by $+y_1$, and

add the resulting eqs, we obtain

$$\boxed{(y_1 y_2'' - y_2 y_1'') + p(t)(y_1 y_2' - y_1' y_2) = 0} \quad (3)$$

Let $W(t) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$$W'(t) = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$W' = y_1 y_2'' - y_2 y_1''$$

Then we can write (3) in the form

$$W' + p(t) W = 0 \quad (\text{seperable})$$

$$\Rightarrow \int \frac{dW}{W} = -\int p(t)\,dt$$

$$\Rightarrow \ln|W| = -\int p(t)\,dt + c_1$$

$$\Rightarrow W = \pm e^{c_1} \cdot e^{-\int p(t)\,dt}$$

$$\Rightarrow W = c\, e^{-\int p(t)\,dt}, \text{ where } c \text{ is constant}$$

Since $e^{-\int p(t)dt} \neq 0$, for all t, then $W(y_1, y_2) \neq 0$ unless $c = 0$ $\boxed{}$

Ex: Find the Wronskian of two solutions of

$$t^2 y'' - t(t+2) y' + (t+2) y = 0, \quad t > 0.$$

Sol.

$$y'' - \frac{t+2}{t} y' + \frac{t+2}{t^2} y = 0$$

$$p(t) = -\left(\frac{t+2}{t}\right) = -\left(1 + \frac{2}{t}\right).$$

$$W(y_1,y_2) = c e^{-\int p(t)\,dt} = c e^{\int (1+\frac{2}{t})\,dt}$$
$$= c e^{t+2\ln t}$$
$$= c t^2 e^t, \quad t>0.$$

Q34) If y_1 and y_2 are a fundamental set of solutions of $t y'' + 2y' + te^t y = 0$ and if $W(y_1,y_2)(1) = 2$, find $W(y_1,y_2)(5)$.

Sol. $y'' + \frac{2}{t} y' + e^t y = 0$.

$$W(y_1,y_2) = c e^{\int \frac{2}{t}\,dt} = c e^{-2\ln|t|} = c t^{-2}$$

$$2 = W(y_1,y_2)(1) = c(1)^{-2} \Rightarrow \boxed{c=2}$$

$$\therefore W(y_1,y_2)(t) = 2t^{-2}$$

$$\Rightarrow W(y_1,y_2)(5) = 2(5)^{-2} = \frac{2}{25}.$$

Rmk (on Abel's thm) Abel's thm give a simple formula for the Wronskian of any pair of solutions of our eq., even if the solutions themselves are not known.

3.4 Repeated Roots, Reduction of order

Reduction of order Method.

Consider the d.e $\boxed{y'' + p(t)y' + q(t)y = 0}$ (*)

Suppose that we know one solution $y_1(t)$ of (*),

To find a second solution for (*), we let

$\boxed{y = v(t)\, y_1(t)}$, then $\boxed{y' = v'y_1 + v y_1'}$

and $y'' = v''y_1 + v'y_1' + v'y_1' + v y_1''$

$$\boxed{y'' = v''y_1 + 2v'y_1' + v y_1''}$$

Substituting for $y, y', + y''$ in Eq (*), we find

that

$$v''y_1 + 2v'y_1' + v y_1'' + p(t)\left[v'y_1 + v y_1'\right] + q(t)(v y_1) = 0$$

or $y_1 v'' + \left(2y_1' + p(t)y_1\right)v' + \cancel{\left(y_1'' + p(t)y_1' + q(t)y_1\right)}^{0} v = 0$

Since y_1 is a solution for (*)

$$\Rightarrow \boxed{y_1 v'' + \left(2y_1' + p(t)y_1\right)v' = 0} \quad (**)$$

let $v' = w$, $v'' = w'$, then (**) becomes

$$y_1 w' + \left(2y_1' + p(t)y_1\right)w = 0$$

$$w' + \left(\frac{2y_1'}{y_1} + p(t)\right) w = 0, \quad y_1 \neq 0.$$

lin. in w.

$$\mu(t) = e^{\int \left(\frac{2y_1'}{y_1} + p(t)\right) dt}$$

$$= e^{2\ln|y_1| + \int p(t)\,dt}$$

$$= y_1^2 e^{\int p(t)\,dt}.$$

$$\therefore\ w(t) = \frac{1}{y_1^2} e^{-\int p(t)\,dt} \left[\int 0 \cdot y_1^2 e^{\int p(t)\,dt}\, dt + c \right]$$

$$w(t) = \frac{c\, e^{-\int p(t)\,dt}}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$$

$$\Rightarrow v' = \frac{W(y_1, y_2)}{y_1^2}$$

$$\Rightarrow v = \frac{y_2}{y_1} = \int \frac{W(y_1, y_2)}{y_1^2}\, dt.$$

$$\Rightarrow \boxed{y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2}\, dt}$$

this is called reduction formula.

Ex. Given that $y_1 = t^2$ is a solution of

$(*)'$ $\boxed{t\, y'' - 6y' + \frac{10}{t} y = 0, \quad t > 0}$. Use

the method of reduction of order to find a second solution of the given d-e.

Sol. Let $\boxed{y = v y_1 = t^2 v(t)}$.

$$\boxed{y' = 2t v + t^2 v'}$$

$$y'' = 2v + 2t v' + 2t v' + t^2 v''$$

$$\boxed{y'' = 2v + 4t v' + t^2 v''}$$

Substitute $y, y', + y''$ into $(*)'$

$$t\left(2v + 4t v' + t^2 v''\right) - 6\left(2t v + t^2 v'\right) + \frac{10}{t}\left(t^2 v\right) = 0$$

$$\Rightarrow \underline{2tv} + 4\underline{t^2 v'} + t^3 v'' \underline{-12tv} - 6\underline{t^2 v'} + \underline{10tv} = 0$$

$$\Rightarrow t^3 v'' - 2t^2 v' = 0$$

$$\Rightarrow t v'' - 2v' = 0, \quad t > 0.$$

Let $v' = w, \quad v'' = w'$

$$t w' - 2w = 0 \Rightarrow w' - \frac{2}{t} w = 0$$

$$\mu(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = t^{-2}, \ t>0.$$

$$w(t) = t^2\left[\int 0 \cdot t^{-2}\,dt + c\right] = ct^2$$

$$\Rightarrow V' = ct^2 \Rightarrow V = \frac{ct^3}{3} + B$$

$$\boxed{V = At^3 + B}, \ A = \frac{c}{3}.$$

$$y = t^2 V = t^2(At^3 + B)$$
$$= At^5 + Bt^2$$

$$\therefore \boxed{y_2 = t^5}$$

Ex: Use the reduction formula to find y_2 in the last example.

Sol: $$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 0, \ t>0.$$

$$W(y_1, y_2) = c\,e^{-\int -\frac{6}{t}\,dt} = c\,e^{6\ln|t|} = ct^6, \ t>0.$$

$$y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2}\,dt = t^2 \int \frac{ct^6}{t^4}\,dt.$$
$$= t^2\left(\frac{t^3}{3}\right) = \frac{c}{3}\underbrace{t^5}_{y_2}.$$

H-W's ① Given that $y_1 = \frac{1}{t}$ is a solution of $2t^2 y'' + 3t y' - y = 0$, $t > 0$. Use the method of reduction of order to find a second solution y_2.

② Given that $y_1 = \frac{\sin x}{\sqrt{x}}$ is one solution of $x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$, $x > 0$

Find a second solution y_2 by using the reduction of order formula.

③ Given that $y_1 = t$ is a solution of

$t^2 y'' + t(t+2) y' + (t+2) y = 0$, $t > 0$.

Find a second solution y_2.

3.5 Nonhomogeneous Equations, Method of Undetermined Coefficients

Consider the nonhomogeneous d.e

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad \text{----} (1)$$

where p, q, g are continuous functions on an open interval I. The corresponding homog. d.e of (1) is $L[y] = y'' + p(t)y' + q(t)y = 0 \quad \text{---}(2)$

Thm 3.5.1 (i) If Y_1 and Y_2 are two solutions of Eq(1), then $Y_1 - Y_2$ is a solution of Eq(2).

(ii) If y_1, y_2 are fundamental set of solutions of Eq(2), then $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.

proof (i) Since Y_1 & Y_2 are solutions of Eq(1), then $L[Y_1] = g(t)$, $L[Y_2] = g(t)$.

$$\Rightarrow L[Y_1 - Y_2] = L[Y_1] - L[Y_2]$$
$$= g(t) - g(t) = 0$$

$\Rightarrow Y_1 - Y_2$ is a solution of Eq (2).

(ii) Since $Y_1 - Y_2$ is a solution of Eq(2) and y_1, y_2 are fundamental set of solutions, then $Y_1 - Y_2$ can be written as a linear combination of y_1 & y_2, i.e., $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$ ∎

Ex. prove that if Y_1, Y_2 are solution of $L[y] = g(t)$, then $\frac{1}{4}Y_1 + \frac{3}{4}Y_2$ is also a solution of $L[y] = g(t)$.

Pf. $L\left[\frac{1}{4}Y_1 + \frac{3}{4}Y_2\right]$

$= \frac{1}{4}L[Y_1] + \frac{3}{4}L[Y_2]$

$= \frac{1}{4}g(t) + \frac{3}{4}g(t)$ since Y_1, Y_2 are solutions of $L[y] = g(t)$.

$= g(t)$

Method of undetermined coefficients

Consider the nonhomog. 2^{nd} order linear d.e

$$ay'' + by' + cy = g(t) \quad ---- (3)$$

where a, b, c are constants; and $g(t)$ is a constant, a poly. function, an exponential function e^{rt}, a sin or cosine function $\sin\beta t$

or $\cos\beta t$, or a finite sums & products of these functions.

Rmk: this method is limited to linear d.e (3), where the conditions on $a, b, c, g(t)$ as above.

Now, to solve Eq (3) by this method, we must do the following

- Find y_h " the solution of the corresponding homog. equation $ay'' + by' + cy = 0$.

- Find any particular solution y_p of the nonhomog. eq (3). Note that y_p depends totally on the form of $g(t)$ as follows.

(a) If $g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ poly., then we let $y_p = t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0)$. and to find $A_0, A_1, \ldots, A_n$ we substitute y_p into Eq (3).

(b) If $g(t) = P_n(t) e^{\alpha t}$, then we let

$$y_p = t^s(A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t}.$$

(c) If $g(t) = P_n(t)\, e^{\alpha t} \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$, then

we let $y_p = t^s\Big[(A_0 + A_1 t + \cdots + A_n t^n)\, e^{\alpha t}\cos\beta t$

$\qquad + (B_0 + B_1 t + \cdots + B_n t^n)\, e^{\alpha t}\sin\beta t\Big]$.

Rmk. Here s is the smallest nonnegative integer ($s = 0, 1$, or 2) that will ensure that no term in y_p is a solution of the corresponding homog. eq.

- The general solution of Eq(3) is

$y_g = y_h + y_p$.

Ex① Find the general solution of the d.e

$$y'' - 3y' - 4y = 3e^{2t}.$$

Sol. step (1) Solve $y'' - 3y' - 4y = 0$.

the aux. eq. $r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$\Rightarrow \boxed{r_1 = 4},\ \boxed{r_2 = -1}$

$$\boxed{y_h = c_1 e^{4t} + c_2 e^{-t}}$$

Step 2 the form of y_p.

$$y_p = Ae^{2t}\cdot t^0 = Ae^{2t}.$$

To find A we substitute y_p into the eq.

$$y_p' = 2Ae^{2t}, \quad y_p'' = 4Ae^{2t}.$$

$$\Rightarrow \cancel{4Ae^{2t}} - 3(2Ae^{2t}) - \cancel{4Ae^{2t}} = 3e^{2t}$$

$$\Rightarrow -6A = 3 \Rightarrow \boxed{A = -\tfrac{1}{2}}$$

$$\therefore \boxed{y_p = -\tfrac{1}{2}e^{2t}}$$

Step 3 $y_g = y_h + y_p$

$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$ is the general solution.

Ex 2. Solve the IVP $\begin{cases} y'' - 3y' - 4y = 3e^{2t} \\ y(0) = 0, \quad y'(0) = 2. \end{cases}$

Sol. From Ex 1, $y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$

$$0 = y(0) = c_1 + c_2 - \tfrac{1}{2} \Rightarrow \boxed{c_1 + c_2 = \tfrac{1}{2}} \cdots (\mathrm{I})$$

$$y' = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}$$

$$2 = y'(0) = 4c_1 - c_2 - 1 \Rightarrow \boxed{4c_1 - c_2 = 3} \cdots (\mathrm{II})$$

$(I)+(II)$: $5c_1 = \frac{7}{2} \Rightarrow \boxed{c_1 = \frac{7}{10}}$

$\boxed{c_2 = -\frac{1}{5}}$

$$\therefore y = \frac{7}{10} e^{4t} - \frac{1}{5} e^{-t} - \frac{1}{2} e^{2t}$$

Ex③. Solve $\boxed{y'' - 3y' - 4y = 2\sin t}$.

Sol. • $y_h = ??$ $r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0 \Rightarrow r = +4, -1$

$$\boxed{y_h = c_1 e^{4t} + c_2 e^{-t}}$$

• $y_p = (A\sin t + B\cos t) \cdot t^0$

$y_p = A\sin t + B\cos t$

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$.

Substitute y_p:

$-A\sin t - B\cos t - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t)$

$= 2\sin t$

$\sin t$: $-A + 3B - 4A = 2 \Rightarrow \boxed{-5A + 3B = 2}$ — (1)

$\cos t$: $-B - 3A - 4B = 0 \Rightarrow \boxed{-3A - 5B = 0}$ (2)

From (1) & (2), $A = -\frac{5}{17}$, $B = \frac{3}{17}$ (أتحقق تفصلهم)

$\therefore y_p = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$

$\therefore y_g = y_h + y_p = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t.$

Ex④. Find the form of y_p (Ex4 – Ex7).

$y'' - 3y' - 4y = -8e^t \cos 2t.$

Sol: $\boxed{y_h = c_1 e^{4t} + c_2 e^{-t}}$

the form of y_p is $y_p = [Ae^t \cos 2t + Be^t \sin 2t] \cdot t^0$

Ex⑤ $y'' - 3y' - 4y = 3e^{2t} + 2\sin t$

$\cdot\, y_h = c_1 e^{4t} + c_2 e^{-t}.$

For y_p we have two subdifferentials

(i) $y'' - 3y' - 4y = 3e^{2t} \Rightarrow y_{p_1} = Ae^{2t}$

(ii) $y'' - 3y' - 4y = 2\sin t \Rightarrow y_{p_2} = (B\sin t + C\cdot\cos t)\cdot t^0$

$y_p = y_{p_1} + y_{p_2} = Ae^{2t} + B\sin t + C\cdot\cos t.$

Ex⑥. $y'' + y = t(1 + \sin t).$

$y'' + y = t + t\sin t$.

$y_h: r^2+1=0 \Rightarrow r=\pm i \Rightarrow \boxed{y_h = c_1\cos t + c_2 \sin t}$

We have two subdifferentials

- $y''+y=t \Rightarrow y_{p_1} = (At+B)\cdot t^0$
- $y''+y=t\sin t \Rightarrow y_{p_2} = [(Ct+D)\sin t + (Et+F)\cos t]\cdot t$

$\therefore y_p = y_{p_1} + y_{p_2}$

$= At + B + (Ct^2+Dt)\sin t + (Et^2+Ft)\cos t$.

<u>Ex7</u>. $y''-y'-2y = \cosh(2t)$

<u>Sol.</u> $y''-y'-2y = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$.

- $y_h: r^2-r-2=0 \Rightarrow (r-2)(r+1)=0$

$r_1=2, \; r_2=-1$

$\boxed{y_h = c_1 e^{2t} + c_2 e^{-t}}$

- For y_p: $y''-y'-2y = \frac{1}{2}e^{2t} \Rightarrow y_{p_1} = Ae^{2t}\cdot t$.

$y''-y'-2y = \frac{1}{2}e^{-2t} \Rightarrow y_{p_2} = Be^{-2t}\cdot t^0$

$y_p = y_{p_1} + y_{p_2} = Ate^{2t} + Be^{-2t}$.

3.6 Variation of parameters

Consider the linear 2^{nd} order d.e

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{---} (1)$$

we have studied the case where p, q are constants and $g(t)$ is one of the functions exp, cos, sin, or poly. or finite sums & products of these functions.

Question How can we solve Eq (1), if g is any function or if p & q are not constants?

Ans. In this case, we use the method of Variation of Parameters

Thm 3.6.1 consider the d.e (1), i.e.,

$$y'' + p(t)y' + q(t)y = g(t).$$

If p, q and $g(t)$ are continuous on an open interval I, and if the functions y_1 and y_2 are fundamental set of solutions of the homog. d.e $y'' + p(t)y' + q(t)y = 0$, then the general solution of the d.e (1) is

$$y_g = y_h + y_p$$

$$= c_1 y_1 + c_2 y_2 + V_1 y_1 + V_2 y_2 \text{, where}$$

$$V_1 = -\int \frac{y_2(t) g(t)}{W(y_1,y_2)(t)} dt \quad , \quad V_2 = \int \frac{y_1(t) g(t)}{W(y_1,y_2)(t)} dt$$

<u>Ex.</u> Find the general solution of the d-e $y'' + 4y = 3\csc t$ by using the method of <u>variation of Parameter</u>.

<u>Sol:</u> • y_h : the aux. eq. is $r^2 + 4 = 0$
$\Rightarrow r = \pm 2i$

$$y_h = c_1 \cos 2t + c_2 \sin 2t$$

let $\boxed{y_1 = \cos 2t}$, $\boxed{y_2 = \sin 2t}$

$$\cdot\ W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

$$= 2\cos^2 2t + 2\sin^2 2t$$

$$= 2(1) = 2 \neq 0$$

$$\cdot\ y_p = V_1 y_1 + V_2 y_2$$

$$= V_1 \cos 2t + V_2 \sin 2t \text{ , where}$$

$$v_1 = -\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)}\, dt = -\int \frac{(\sin 2t)(3\csc t)}{2}\, dt$$

$$= -3\int \frac{\cancel{2}\sin t \cos t}{\cancel{2}} \cdot \csc t \, dt$$

$$= -3\int \cos t \, dt = -3\sin t .$$

$$\therefore \boxed{v_1 = -3\sin t}$$

$$v_2 = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)}\, dt = \int \frac{(\cos 2t)(3\csc t)}{2}\, dt$$

$$= \frac{3}{2}\int (1 - 2\sin^2 t) \csc t \, dt$$

$$= \frac{3}{2}\int (\csc t - 2\sin t)\, dt$$

$$= \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t$$

$$\therefore y_p = v_1 y_1 + v_2 y_2$$

$$= -3\sin t \cos 2t + \left(\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right) \cdot \sin 2t$$

$$= -3\sin t \cos 2t + 3\cos t \sin 2t + \frac{3}{2}\sin 2t \ln|\csc t - \cot t|$$

$$= 3\sin t + \frac{3}{2}\sin 2t \ln|\csc t - \cot t|$$

$$\Rightarrow y_g = y_h + y_p$$

$= c_1 \cos 2t + c_2 \sin 2t + 3\sin t + \frac{3}{2}\sin 2t \ln|\csc t - \cot$

Ex. Solve the following d.e

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0.$$

Sol. y_h : $x^2y'' - 3xy' + 4y = 0.$

this is Euler Eq.

Let $t = \ln x$. $(A = 1, B = -3, C = 4)$.

The d.e becomes

$$\frac{d^2y}{dt^2} + (-3-1)\frac{dy}{dt} + 4y = 0$$

or $$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y.$$

The aux. eq. is $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0 \Rightarrow r = 2, 2.$$

$$y_h = c_1 e^{2t} + c_2 t e^{2t}$$

$$= c_1 e^{2\ln x} + c_2 (\ln x) e^{2\ln x}$$

$$= c_1 x^2 + \underline{c_2 x^2 \ln x}$$

$\boxed{y_1 = x^2}$, $\boxed{y_2 = x^2 \ln x}$

- standard $\quad y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x, x>0$

$g(x) = \ln x$.

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix}$$

$$= x^3 + 2x^3\ln x - 2x^3\ln x$$

$$= x^3 \neq 0 \quad \text{since } x > 0.$$

$y_p = v_1 y_1 + v_2 y_2 = v_1 x^2 + v_2 x^2 \ln x,$

where $v_1 = -\int \frac{y_2(x)g(x)}{W}dx$

$$= -\int \frac{(x^2\ln x)(\ln x)}{x^3}dx$$

$$= -\int \frac{(\ln x)^2}{x}dx \qquad \text{let } u = \ln x, \; du = \frac{1}{x}dx$$

$$= -\frac{(\ln x)^3}{3}.$$

$$v_2 = \int \frac{y_1(x)g(x)}{W}dx = \int \frac{x^2 \cdot \ln x}{x^3}dx$$

$$= \int \frac{\ln x}{x}dx \qquad u = \ln x, \; du = \frac{1}{x}dx$$

$$= \frac{(\ln x)^2}{2}.$$

$$\therefore y_p = \frac{-(\ln x)^3}{3} \cdot x^2 + \frac{(\ln x)^2}{2} \cdot (x^2 \ln x)$$

$$= \left(-\frac{1}{3} + \frac{1}{2}\right) x^2 (\ln x)^3 = \frac{1}{6} x^2 (\ln x)^3 .$$

$$\therefore y_g = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3 .$$

<u>H.w's</u> ① Find the general solution of

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

② Given that $y_1 = \frac{\sin x}{\sqrt{x}}$, $y_2 = \frac{\cos x}{\sqrt{x}}$

are solutions of the homog. Eq.

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0, \quad x > 0 .$$

Find y_p of the nonhomog. eq.

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 3 x^{3/2} \sin x, \quad x > 0$$

CH4 Higher Order linear Equations

4.1 General theory of nth order linear eqs.

A nth order linear d-e is an equation of the form

$$\boxed{L[y] = y^{(n)} + a_{n-1}(t)\,y^{(n-1)} + \cdots + a_1(t)\,y' + a_0 y = g(t)} \quad (1)$$

with corresponding homogeneous D.E

$$\boxed{L[y] = y^{(n)} + a_{n-1}(t)\,y^{(n-1)} + \cdots + a_1(t)\,y' + a_0(t)\,y = 0} \quad (2)$$

· Eq(1) requires n intial Conditions

$$\boxed{y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}} \quad (3)$$

Thm 4.1.1 If the functions $a_{n-1}(t), \ldots, a_1'(t), a_0(t)$, and g are Continuous on the open interval $I = (\alpha, \beta)$, then there exists exactly one solution $y = \phi(t)$ of the D.E(1) that also satisfies the initial conditions (3), where t_0 is any point in I.

Ex. Determine the interval in which the solution of the following IVP is certain to exist.

$$\begin{cases} (x-1)\, y^{(4)} + (x+1)\, y'' + (\tan x)\, y = 0 \\ y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0 \end{cases}$$

- The general solution for the homog. eq (2) is given by

$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$, where $y_1, y_2, \ldots, y_n$ are solutions of Eq (2) and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

To find $c_1, \ldots, c_n$, we use the initial conditions given in (3).

<u>Thm 4.1.2</u> If the function $a_0, a_1, \dots, a_{n-1}$ are continuous on an open interval $I = (\alpha, \beta)$, if the functions $y_1, \dots, y_n$ are solutions of (2) and if

$W(y_1, y_2, \dots, y_n)(t_0) \neq 0$ for some $t_0 \in I$,

then every solution of Eq (2) can be expressed as a linear combination of $y_1, y_2, \dots, y_n$.

- A set of solutions $y_1, y_2, \dots, y_n$ of Eq (2) whose Wronskian is nonzero is reffered to as a <u>fundamental set of solutions</u>

<u>Ex.</u> show that $\{1, t, t^3\}$ form a fundamental set of solutions for the D.E

$$t y^{(3)} - y'' = 0$$

<u>Sol.</u> let $y_1 = 1$, $y_2 = t$, $y_3 = t^3$

(i) $y_1' = 0,\ y_1'' = 0,\ y_1''' = 0$

$L.H.S = t y^{(3)} - y'' = t(0) - 0 = 0.$

$\therefore y_1 = 1$ is a solution.

$y_2' = 1,\ y_2'' = y_2''' = 0.$

$L.H.S = t(0) - 0 = 0.$

$\therefore y_2 = t$ is a solution.

$y_3' = 3t^2,\ y_3'' = 6t,\ y_3''' = 6.$

$L.H.S = t y^{(3)} - y'' = t(6) - 6t = 0$

$\therefore y_3 = t^3$ is a solution.

(ii) $W(1, t, t^3) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t$

$W(1, t, t^3)(1) = 6(1) = 6 \neq 0$

• Linear Dependence and Independence

Df. (1) A functions $f_1, f_2, \dots, f_n$ are linearly independent if

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$$

(2) A functions $f_1, \dots, f_n$ are linearly dependent if there exists $c_1, \dots, c_n$ not all zero such that

$$c_1 f_1 + \dots + c_n f_n = 0.$$

Ex. $f_1 = 1$, $f_2 = 2+t$, $f_3 = 3-t^2$ are linearly independent. Indeed,

let $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$$c_1(1) + c_2(2+t) + c_3(3-t^2) = 0$$

$$(c_1 + 2c_2 + 3c_3) + c_2 t - c_3 t^2 = 0$$

$$\Rightarrow \quad c_1 + 2c_2 + 3c_3 = 0, \quad c_2 = 0, \quad -c_3 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow \text{lin. indep.}$$

Ex. Determine whether the functions $f_1(t)=1$, $f_2(t)=t+2$, $f_3(t)=3-t^2$, $f_4(t)=t^2+4t$ are linearly independent or dependent on any interval I.

Sol. Let $k_1 f_1(t)+k_2 f_2(t)+k_3 f_3(t)+k_4 f_4(t)=0$

$$\Rightarrow k_1 \cdot 1+k_2(t+2)+k_3(3-t^2)+k_4(t^2+4t)=0$$

constants terms : $k_1+2k_2+3k_3=0$ —— (1)

t : $k_2+4k_4=0$ —— (2)

t^2 : $-k_3+k_4=0$ —— (3)

These three equations with four unknowns, have many solutions.

Since, if we set $\boxed{k_4=t} \xrightarrow{eq(3)} \boxed{k_3=t}$

$\xrightarrow{eq(2)} \boxed{k_2=-4t}$

$eq(1) \Rightarrow k_1-8t+3t=0$

$\Rightarrow \boxed{k_1=5t}$, $t\in\mathbb{R}$

Thus, $\{f_1, f_2, f_3, f_4\}$ are lin. dep. on every interval.

4.2 Homogeneous Equations with constant coefficients.

Consider the nth order linear homog. d.e $L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0 \quad \cdots (1)$

where $a_0, a_1, \cdots, a_n$ are constants & $a_n \neq 0$.

To solve Eq (1), we use the same our knowledge of 2^{nd} order linear d.e's considered in CH3.

__Ex ①__ Solve $2y''' - 4y'' - 2y' + 4y = 0$.

the aux. eq. is $2r^3 - 4r^2 - 2r + 4 = 0$

$\Rightarrow 2r^2(r-2) - 2(r-2) = 0$

$(r-2)(2r^2-2) = 0 \Rightarrow r = 2, 1, -1$

$y_h = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}$.

__Ex ②__ Solve $y''' - 5y'' + 3y' + y = 0$

The aux. eq. is $r^3 - 5r^2 + 3r + 1 = 0$

factors of 1 are ± 1

$(1)^3 - 5(1)^2 + 3(1) + 1 = 1 - 5 + 3 + 1 = 0$

$\Rightarrow$ 1 is a zero (i.e, $r-1$ is a factor)

$$\begin{array}{r|l} & r^2-4r-1 \\ \hline r-1 & r^3-5r^2+3r+1 \\ & -r^3 \pm r^2 \\ \hline & -4r^2+3r \\ & \pm 4r^2 \mp 4r \\ \hline & -r+1 \\ & \pm r \mp 1 \\ \hline & 0 \end{array}$$

$\therefore$ The aux. eq. is $(r-1)(r^2-4r-1)=0$

$$\Rightarrow r = 1, \quad \frac{4 \pm \sqrt{16-4(1)(-1)}}{2(1)}$$

$$= 1, \quad 2 \pm \sqrt{5}.$$

$$y_h = c_1 e^t + c_2 e^{(2+\sqrt{5})t} + c_3 e^{(2-\sqrt{5})t}.$$

Ex. 3. $y^{(4)} - 2y'' + 3y' - 2y = 0$

The aux. eq. is $r^4 - 2r^2 + 3r - 2 = 0$

$(1)^4 - 2(1)^2 + 3(1) - 2 = 1 - 2 + 3 - 2 = 0$

$\boxed{r=1}$ is a root $\Rightarrow r-1$ is a factor

$$\begin{array}{r|l} & r^3+r^2-r+2 \\ \hline r-1 & r^4-2r^2+3r-2 \\ & -r^4 \pm r^3 \\ \hline & r^3-2r^2+3r-2 \\ & -r^3 \pm r^2 \\ \hline & -r^2+3r-2 \\ & \pm r^2 \mp r \\ \hline & 2r-2 \\ & -2r \pm 2 \\ \hline & 0 \end{array}$$

$$(r-1)\underbrace{(r^3+r^2-r+2)}_{\downarrow} = 0 .$$

$r=-2$ is a root

$r+2$ is a factor

$$\begin{array}{r|rrrr} & 1 & 1 & -1 & 2 \\ & & -2 & 2 & -2 \\ -2 & & & & \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$$(r-1)(r+2)(r^2-r+1) = 0$$

$$r=1, \quad r=-2, \quad r = \frac{1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore y_h = c_1 \underbrace{e^{t}}_{y_1} + c_2 \underbrace{e^{-2t}}_{y_2} + c_3 \underbrace{e^{\frac{1}{2}t} \cos \tfrac{\sqrt{3}}{2} t}_{y_3} + c_4 e^{\frac{1}{2}t} \underbrace{\sin \tfrac{\sqrt{3}}{2} t}_{y_4} .$$

H.w's

Ex. 4 Solve $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

Ex. 5 Solve $y^{(4)} + y = 0$

Ex. 6 Solve $y^{(5)} + 3y^{(4)} - 5y''' + 17y'' - 36y' + 20y = 0$

4.3 The method of Undetermined Coefficients

Consider the nth order linear nonhomogeneous eq. with constant coefficients

$$L[y] = a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(t) \quad (1)$$

we still use the method of undetermined coefficient to find y_p if g is constant, sin, cos, exp., poly, finite sums & products of these functions as we did in sec. 3.5

Ex. Solve the following d.e

① $y''' - 3y'' + 3y' - y = 4e^t$.

y_h: the aux. eq. is $r^3 - 3r^2 + 3r - 1 = 0$

$$r^3 - 1 - 3r^2 + 3r = 0$$

$$(r-1)(r^2+r+1) - 3r(r-1) = 0$$

$$(r-1)(r^2+r+1-3r) = 0$$

$$(r-1)(r^2-2r+1) = 0 \Rightarrow (r-1)^3 = 0$$

$$r = 1, 1, 1.$$

$$y_h = c_1 e^t + c_2 t e^t + c_3 t^2 e^t .$$

The form of y_p is $y_p = A e^t \cdot t^3 = A t^3 e^t$.

To find y_p we substitute it into the eq. - - - -

② $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

y_h: $r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2+1)^2 = 0 \Rightarrow r = \pm i, \pm i.$

$$y_h = c_1 \cos t + c_2 \sin t + (c_3 \cos t + c_4 \sin t)t .$$
$$= (c_1 + c_3 t)\cos t + (c_2 + c_4 t)\sin t$$

the form of y_p is $y_p = (A\sin t + B\cos t).t^2.$

$$\Rightarrow y_p = At^2 \sin t + Bt^2 \cos t$$

③ $y''' - 4y' = t + 3\cos t + e^{-2t}$

y_h: $r^3 - 4r = 0 \Rightarrow r(r^2-4) = 0 \Rightarrow r = 0, \pm 2$

$$y_h = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

To find the form of y_p, we have 3 sub diff's.

$y''' - 4y' = t \Rightarrow y_{p_1} = (At+B).t = At^2 + Bt.$

$y''' - 4y' = 3\cos t \Rightarrow y_{p_2} = (C\sin t + D\cos t).t^0$

$y''' - 4y' = e^{-2t} \Rightarrow y_{p_3} = Ee^{-2t}.t = Ete^{-2t}$

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3}$

$$= At^2 + Bt + C\sin t + D\cos t + Ete^{-2t}.$$

- - - -

④ $y^{(5)} + 4y''' = \cos^2 t - \sin^2 t$.

$y^{(5)} + 4y''' = \cos(2t)$.

y_h: $r^5 + 4r^3 = 0 \Rightarrow r^3(r^2+4) = 0$

$\Rightarrow r = 0, 0, 0, \pm 2i$

$y_h = c_1 + c_2 t + c_3 t^2 + c_4 \cos 2t + c_5 \sin 2t$.

$y_p = (A\cos 2t + B\sin 2t) \cdot t$

$= At\cos 2t + Bt\sin 2t$.

....

⑤ $y^{(4)} + y = t$

y_h: $r^4 + 1 = 0 \Rightarrow r^4 + 2r^2 + 1 = 2r^2$

$(r^2+1)^2 = 2r^2$

$(r^2+1)^2 - (\sqrt{2}r)^2 = 0$

$\Rightarrow (r^2 - \sqrt{2}r + 1)(r^2 + \sqrt{2}r + 1) = 0$

$\Rightarrow r = \dfrac{\sqrt{2} \pm \sqrt{2-4}}{2}, \dfrac{-\sqrt{2} \pm \sqrt{2-4}}{2}$

$= \dfrac{\sqrt{2} \pm \sqrt{2}i}{2}, \dfrac{-\sqrt{2} \pm \sqrt{2}i}{2}$

$= \dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}}i, -\dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}}i$

$$y_h = c_1 e^{\frac{1}{\sqrt{2}}t}\cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 e^{\frac{1}{\sqrt{2}}t}\sin\left(\frac{1}{\sqrt{2}}t\right)$$

$$+ c_3 e^{-\frac{1}{\sqrt{2}}t}\cos\left(\frac{1}{\sqrt{2}}t\right) + c_4 e^{-\frac{1}{\sqrt{2}}t}\sin\left(\frac{1}{\sqrt{2}}t\right)$$

$$y_p = At + B \;,\; y_p' = A \;,\; y_p'' = y_p''' = y_p^{(4)} = 0$$

<u>Substitute</u> $0 + At + B = t \Rightarrow A = 1,\ B = 0$

$$\therefore y_p = t .$$

$$\therefore y_g = y_h + y_p = \cdots\cdots$$

⑥ $y^{(4)} + y''' = 1 - t^2 e^{-t}$.

y_h : $r^4 + r^3 = 0 \Rightarrow r^3(r+1) = 0$

$r = 0, 0, 0, -1$

$$y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} .$$

For y_p, we have $y^{(4)} + y''' = 1 \Rightarrow y_{p_1} = A \cdot t^3$

$y^{(4)} + y''' = -t^2 e^{-t}$

$y_{p_2} = (Bt^2 + Ct + D)\, e^{-t} \cdot t$

$$\therefore y_p = y_{p_1} + y_{p_2} = At^3 + (Bt^3 + Ct^2 + Dt)\, e^{-t}$$

CH5 Series solutions of second order linear equations

5.1 Review of power series:

In this chapter, we discuss the use of power series to construct fundamental sets of solutions y_1 and y_2 of second order linear d.e's whose coefficients are function of the independent variable, and we write the solutions y_1 and y_2 in terms of power series.

- Summerizing some results about power series that we need.

① A power series about the point x_0 "center" has the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and it is said to converge at x if

$$\lim_{k\to\infty} \sum_{n=0}^{k} a_n (x-x_0)^n \text{ exists for}$$

that x.

[2] the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is said to converge absolutely at a point x if the series $\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$ converges.

[3] To test the absolute convergence for the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ we use the ratio test.

• Ratio Test $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right|$$

$$= |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= |x-x_0| \cdot L,$$

then the power series converges absolutely if $|x-x_0| \cdot L < 1$ and diverges if

$|x-x_0|\cdot L > 1$. If $|x-x_0|\cdot L = 1$, then the test is inconclusive.

ex. For which values of x does the power series $\sum_{n=1}^{\infty}(-1)^{n+1} n(x-2)^n$ converge?

Sol. $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}\, n\,(x-2)^n}\right|$

$= |x-2| \lim_{n\to\infty} \frac{n+1}{n}$

$= |x-2|\cdot 1 < 1$

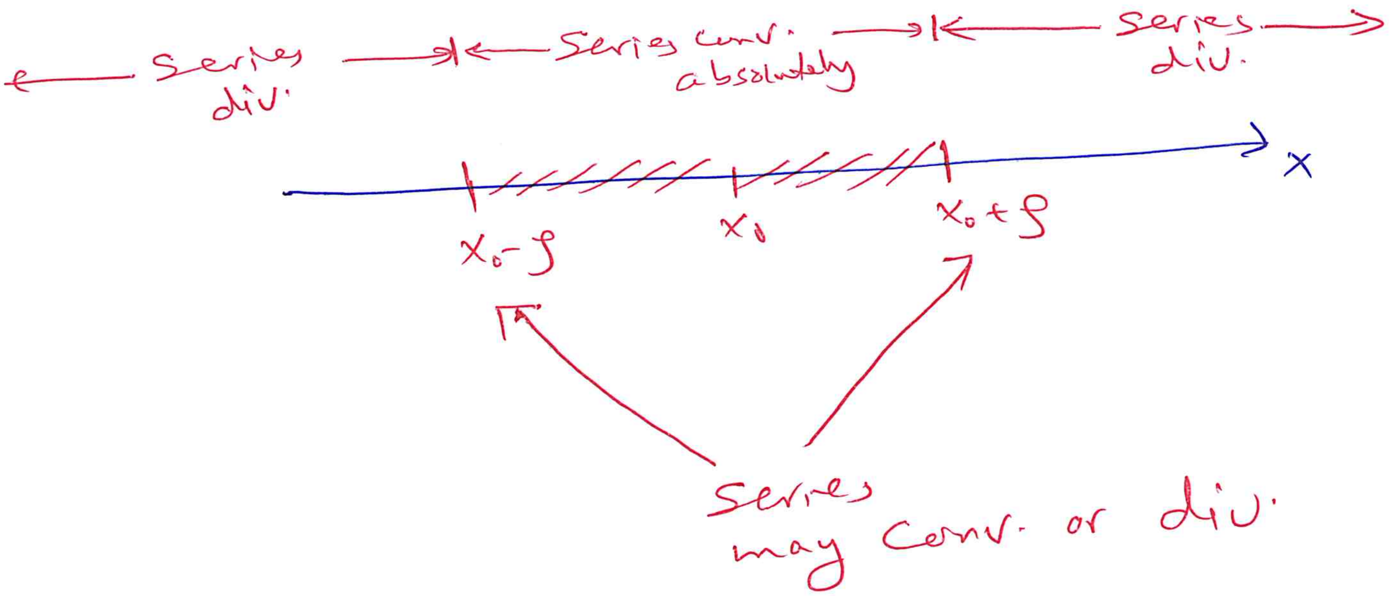
$\Rightarrow -1 < x-2 < 1 \Rightarrow \boxed{1 < x < 3}$

$\boxed{x=1}$, $\sum_{n=1}^{\infty}(-1)^{n+1} n(-1)^n = \sum_{n=1}^{\infty} -n$ div.

$\boxed{x=3}$ $\sum_{n=1}^{\infty}(-1)^{n+1} n(1)^n = \sum(-1)^{n+1} n$ div.

$\therefore$ The interval of absolute convergence is $(1,3)$.

$\boxed{4}$ The radius of convergence is a positive number ρ such that $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. The interval $|x-x_0| < \rho$ is called the interval of convergence.



(The interval of convergence of a power series).

Ex. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$.

Sol. $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{n(x-2)^n(-1)^{n+1}} \right|$

$= |x-2| \lim_{n\to\infty} \frac{n+1}{n}$

$= |x-2| < 1$

center ρ = radius.

The radius of convergence $= \rho = 1$.

5 Differentiation and Integration of a power series

If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

and so on

$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} + C.$$

$\boxed{6}$ The Taylor series for the function f about $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

A function f that has a Taylor series expansion about $x = x_0$ with radius of convergence $\rho > 0$ is said to be <u>analytic</u> at $x = x_0$, like $\sin x$, $\cos x$, e^x, --

$\boxed{7}$ <u>Shifting of Index of Summation</u>

<u>Ex.</u> Write the series $\sum_{n=2}^{\infty} (n+2)(n+1) a_n (x-1)^{n-2}$ as a series involves $(x-1)^n$.

<u>Sol.</u> $$\sum_{n=2}^{\infty} (n+2)(n+1) a_n (x-1)^{n-2}$$
$$= \sum_{n=0}^{\infty} (n+4)(n+3) a_{n+2} (x-1)^n$$

Ex: write the given expression as a single sum involves x^n.

$\boxed{1}$ $$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + 2a_n\right] x^n.$$

$\boxed{2}$ $$x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} \left[n(n+1) a_{n+1} + a_n\right] x^n.$$

5.2 Series Solutions Near an ordinary points, Part I.

In Ch3, we described methods of solving 2nd order linear d.e with constant coefficients. We now consider methods of solving 2nd order linear d.e when the coefficients are functions of the independent variables. It is sufficient to consider the homogeneous eq.

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad ---- (1)$$

since the procedure for the corresponding nonhomog. Eq. is similar.

Df: A point x_0 such that $P(x_0) \neq 0$ in Eq (1) is called an ordinary point. If $P(x_0) = 0$, then x_0 is called singular point.

We assume that P, Q, & R in Eq (1) are continuous. It follows that there is an interval about x_0 in which $P(x)$ is never zero. In that interval, Eq (1) can be written $y'' + p(x)y' + q(x)y = 0$ ---- (2)

where $p(x) = \dfrac{Q(x)}{P(x)}$, $q(x) = \dfrac{R(x)}{P(x)}$ are

Continuous functions. therefore, by thm 3.2.1, there exists a unique solution that satisfies Eq (1) together with the interval conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$. To find such Solution in terms of power series, we assume such a solution has the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

defined on an interval of convergence $|x - x_0| < \rho$, where $\rho > 0$ is the radius of convergence and x_0 is ordinary point.

Ex. Find ordinary and singular points of

① $(x^2 - x)y'' + xy' - 2x^2 y = 0$

Sol. $P(x) = x^2 - x = x(x-1) = 0 \Rightarrow x = 0, x = 1$

are singular points. All other points real or complex are ordinary.

② $(x^2+4)y'' + xy = 0$

$P(x) = x^2+4 = 0 \Rightarrow x = \pm 2i$ are singular pts.

All other pts real or complex are ordinary.

Ex. Find a series solution of

$$y'' + y = 0, \quad -\infty < x < \infty.$$

Sol. $P(x) = 1 \neq 0$ for all x, then every point is an ordinary pt, so we choose $x_0 = 0$ as a simplest choice.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y, y', y'' into Eq.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+\cancel{2}-\cancel{2}} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \Big((n+2)(n+1) a_{n+2} + a_n\Big) x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0, \quad \forall n = 0, 1, 2, \ldots$$

or
$$\boxed{a_{n+2} = -\frac{1}{(n+2)(n+1)}\, a_n \,, \quad n = 0, 1, 2, \ldots}$$

$n=0$: $\quad a_2 = \dfrac{-1}{(2)(1)} a_0 = \dfrac{-1}{2!} a_0$

$n=1$: $\quad a_3 = -\dfrac{1}{(3)(2)} a_1 = -\dfrac{a_1}{3!}$

$n=2$: $\quad a_4 = \dfrac{-1}{(4)(3)} a_2 = \dfrac{-1}{(4)(3)}\left(\dfrac{-1}{2!} a_0\right)$
$$= \frac{1}{4!} a_0 .$$

$n=3$: $\quad a_5 = \dfrac{-1}{(5)(4)} a_3 = \dfrac{-1}{(5)(4)}\left(-\dfrac{a_1}{3!}\right)$
$$= \frac{a_1}{5!}$$

$$\therefore\ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots$$
$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= a_0 \cos x + a_1 \sin x .$$

<u>Ex.</u> Find two linearly independent power series solutions y_1 and y_2 of Airy's Eq.
$$y'' - xy = 0, \quad -\infty < x < \infty .$$
about the ordinary point $x_0 = 0$.

Sol. Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute: $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow (2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$\Rightarrow 2a_2 = 0$ and $(n+2)(n+1) a_{n+2} - a_{n-1} = 0, \quad n = 1, 2, \ldots$

or $\boxed{a_2 = 0}$, $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, \ldots$

$\boxed{n=1}$ $a_3 = \dfrac{a_0}{(3)(2)} = \dfrac{a_0}{6}$

$\boxed{n=2}$ $a_4 = \dfrac{a_1}{(4)(3)} = \dfrac{a_1}{12}$

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$\boxed{n=3}$ $a_5 = \frac{a_2}{(5)(4)} = 0$

$\boxed{n=4}$ $a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{6(6)(5)} = \frac{a_0}{180}$.

$\boxed{n=5}$ $a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(12)(42)} = \frac{a_1}{504}$

$\vdots$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + 0x^2 + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + 0x^5 + \frac{a_0}{180}x^6 + \frac{a_1}{504}x^7 + \cdots$$

$$= a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right)$$

$= a_0 y_1 + a_1 y_2$, where

$$y_1 = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots$$

$$y_2 = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots$$

Next, we need to prove that y_1 & y_2 are linearly indep. and hence form a fundamental set of solutions, we combute $W(y_1, y_2)(0)$.

$$W(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Ex. consider the d.e

$$(x^2+1)y'' - 4xy' + 6y = 0 \quad , \quad -\infty < x < \infty .$$

Find two series solutions y_1 and y_2 near an ordinary point $x_0 = 0$. Show that y_1 and y_2 form a fundamental set of solutions.

Sol. let $y = \sum_{n=0}^{\infty} a_n (x-0)^n = a_0 + a_1(x-0) + a_2(x-0)^2 + \cdots$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute $(x^2+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\Rightarrow (2)(1) a_2 x^0 + (3)(2) a_3 x - 4(1) a_1 x + 6 a_0 x^0 + 6 a_1 x + \sum_{n=2}^{\infty} \left[n(n-1) a_n + (n+2)(n+1) a_{n+2} - 4n a_n + 6 a_n \right] x^n = 0$$

$$\Rightarrow 2a_2 + 6a_0 + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty} \left[(n^2 - n - 4n + 6) a_n + (n+2)(n+1) a_{n+2} \right] x^n = 0$$

$\Rightarrow \; 2a_2 + 6a_0 = 0 \; , \; 6a_3 + 2a_1 = 0$

$(n^2 - 5n + 6)\, a_n + (n+2)(n+1)\, a_{n+2} = 0 \; , \; n = 2, 3, 4, \ldots$

$\Rightarrow \; \boxed{a_2 = -3a_0} \; , \; \boxed{a_3 = -\frac{1}{3} a_1}$

$$a_{n+2} = -\frac{(n-3)(n-2)}{(n+2)(n+1)} \; , \; n = 2, 3, 4, \ldots$$

$\boxed{n = 2}$ $\quad a_4 = 0$

$\boxed{n = 3}$, $\quad a_5 = 0 \; \ldots\ldots \; a_n = 0, \; \forall n = 4, 5, 6, \ldots$

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cancel{(a_4 x^4 + \ldots)}^{\,0}$

$= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3$

$= a_0 (1 - 3x^2) + a_1 (x - \frac{1}{3} x^3)$

$= a_0 y_1 + a_1 y_2$, where

$y_1 = 1 - 3x^2 \; , \; y_2 = x - \frac{1}{3} x^3$

$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow \{y_1, y_2\}$ are lin. indep.

hence they form a fundamental set of solutions

<u>H.W</u> $(x^2 - 1) y'' - 6xy' + 12y = 0$ about $x_0 = 0$

Ans: $y_1 = 1 + 6x^2 + x^4$, $y_2 = x + x^3$

5.3 Series Solutions Near an ordinary point part II.

In the last section we learned how to find a power series solution of $\boxed{P(x)y''+Q(x)y'+R(x)y=0}$ (1)

where $P(x)$, $Q(x)$ & $R(x)$ are polynomials in the neighborhood of an ordinary point x_0.

(i.e, $P(x_0)\neq 0$). Hence we can write Eq(1)

as $\boxed{y''+p(x)y'+q(x)y=0}$ (2)

where $p(x)=\dfrac{Q(x)}{P(x)}$, $q(x)=\dfrac{R(x)}{P(x)}$ are

analytic functions (i.e, p and q have Taylor expansion about x_0 that converges to $p(x)$ and $q(x)$ respectively in the interval $|x-x_0|<\rho$, where $\rho>0$. i.e, $p(x)=\sum_{n=0}^{\infty} p_n(x-x_0)^n = p_0+p_1(x-x_0)+p_2(x-x_0)^2 \cdots$

$$q(x)=\sum_{n=0}^{\infty} q_n(x-x_0)^n = q_0+q_1(x-x_0)+q_2(x-x_0)^2+\cdots$$

Now assuming that Eq(1) does have a solution y has a Taylor series $\boxed{y=\sum_{n=0}^{\infty} a_n(x-x_0)^n}$ (3)

that converges for $|x-x_0|<\rho$, $\rho>0$, we found that a_n can be determined by substituting the series (3) into Eq(1).

<u>claim</u> $\quad y^{(m)}(x_0) = m!\, a_m$.

<u>Pf (Claim)</u>. $y' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$

$$= a_1 + 2a_2(x-x_0) + \cdots$$

$$\Rightarrow y'(x_0) = a_1 = 1!\, a_1$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-x_0)^{n-2}$$

$$= 2a_2 + 3(2)\, a_3(x-x_0) + \cdots$$

$$\Rightarrow y''(x_0) = 2a_2 = 2!\, a_2$$

$$y'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)\, a_n (x-x_0)^{n-3}$$

$$= 3(2)(1)a_3 + 4(3)(2)\, a_4 (x-x_0) + \cdots$$

$$\Rightarrow y'''(x_0) = 6a_3 = 3!\, a_3$$

$$\vdots$$

$$y^{(m)}(x_0) = m!\, a_m.$$

Ex 1. Suppose that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of the IVP $\begin{cases} y'' + e^x y = 0 \\ y(0) = 1, \; y'(0) = 1 . \end{cases}$

Find a_0, a_1, a_2, a_3, a_4. Then write the solution.

Sol. $a_0 = y(0) = 1, \quad a_1 = \dfrac{y'(0)}{1!} = 1$

$a_2 = \dfrac{y''(0)}{2!}$. Now from the eq. $\boxed{y'' = -e^x y}$

$$\Rightarrow y''(0) = -e^0 y(0) = -1(1) = -1$$

$$\therefore a_2 = \frac{y''(0)}{2!} = \frac{-1}{2}$$

$$\boxed{y'''_{(x)} = -e^x y - e^x y'}$$

$$y'''(0) = -e^0 y(0) - e^0 y'(0)$$

$$= -1 - 1 = -2$$

$$\therefore a_3 = \frac{y'''(0)}{3!} = \frac{-2}{3!} = \frac{-1}{3}.$$

$$y^{(4)} = -e^x y - e^x y' - e^x y' - e^x y''$$

$$= -e^x y - 2e^x y' - e^x y''$$

$$y^{(4)}(0) = -y(0) - 2y'(0) - y''(0)$$

$$-\cancel{1} - 2 + \cancel{1} = -2$$

$$a_4 = \frac{y^{(4)}(0)}{4!} = \frac{-2}{4!} = \frac{-2}{24} = \frac{-1}{12}$$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \cdots$$

Ex 2. Consider the IVP

$$\begin{cases} 4y'' - e^x y' + 2y\cos x = 0 \\ y(0) = 3 \quad , \ y'(0) = 2 . \end{cases}$$

Find the first four nonzero terms of the series solution.

Sol. $p(x) = 4 \neq 0 \Rightarrow$ All points are ordinary.

Take $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$a_0 = y(0) = 3 \quad , \ a_1 = y'(0) = 2$$

$$y'' = \frac{e^x}{4} y' - \frac{1}{2} y \cos x$$

$$y''(0) = \frac{1}{4} y'(0) - \frac{1}{2} y(0) \cos 0$$
$$= \frac{1}{4}(2) - \frac{1}{2}(3)(1) = -1$$

$$a_2 = \frac{y''(0)}{2!} = \frac{-1}{2}.$$

$$y'''(x) = \frac{1}{4}e^x y' + \frac{1}{4}e^x y'' - \frac{1}{2}y' \cos x + \frac{1}{2}y \sin x$$

$$y'''(0) = \frac{1}{4}y'(0) + \frac{1}{4}y''(0) - \frac{1}{2}y'(0) + 0$$

$$= \frac{1}{4}(2) + \frac{1}{4}(-1) - \frac{1}{2}(2) = -\frac{3}{4}$$

$$a_3 = \frac{y'''(0)}{3!} = \frac{-\frac{3}{4}}{6} = -\frac{1}{8}.$$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= 3 + 2x - \frac{1}{2}x^2 - \frac{1}{8}x^3 + \cdots$$

__Ex 3.__ Find a power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ for the equation

$$y'' - (x^3 + 3x + 2)y' + 3\cos(2x)\, y = 0$$

__Sol.__ $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_0 = y(0)$, $a_1 = y'(0)$

$$\boxed{y'' = (x^3 + 3x + 2)y' - 3y\cos(2x)}$$

$$y''(0) = 2y'(0) - 3y(0)$$

$$= 2a_1 - 3a_0$$

$$a_2 = \frac{y''(0)}{2!} = \frac{2a_1 - 3a_0}{2!} = a_1 - \frac{3}{2}a_0$$

$$y'''(x) = (3x^2+3)y' + (x^3+3x+2)y'' - 3y'\cos(2x) + 6y\sin 2x$$

$$y'''(0) = \cancel{3y'(0)} + 2y''(0) \cancel{- 3y'(0)}$$

$$= 2(2a_1 - 3a_0)$$

$$= 4a_1 - 6a_0$$

$$a_3 = \frac{y'''(0)}{3!} = \frac{4a_1 - 6a_0}{6} = \frac{2}{3}a_1 - a_0$$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + \left(a_1 - \frac{3}{2}a_0\right)x^2 + \left(\frac{2}{3}a_1 - a_0\right)x^3 + \cdots$$

$$= a_0\left(1 - \frac{3}{2}x^2 - x^3 + \cdots\right) + a_1\left(x + x^2 + \frac{2}{3}x^3 + \cdots\right)$$

$= a_0 y_1 + a_1 y_2$, where

$$y_1 = 1 - \frac{3}{2}x^2 - x^3 + \cdots$$

$$y_2 = x + x^2 + \frac{2}{3}x^3 + \cdots$$

Thm 5.3.1 If x_0 is an ordinary point of the d.e (1) : $P(x)y'' + Q(x)y' + R(x)y = 0$. The general solution of Eq (1) is

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0, a_1 are arbitrary and y_1, y_2 are two power series solutions that are analytic at x_0. The solutions y_1 & y_2 form a fundamental set of solutions. Further, the radius of convergence for each of the series y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p & q.

Rmk. thm 5.3.1 is general thm i.e., $P, Q, + R$ can be Poly. or not.

Ex. what is the radius of convergence of the Taylor series for $f(x) = \dfrac{1}{x^2-2x-3}$ about $x_0 = 2$

Sol: Singular points: $x^2-2x-3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$x = 3, \ x = -1$

$\rho_1 = \text{dist.}(x_0, 3) = \text{dist.}(2,3) = 1$

$\rho_2 = \text{dist.}(x_0, -1) = \text{dist.}(2,-1) = 3$

the radius of convergence $\rho = \min\{\rho_1, \rho_2\} = 1$
and the series conv. for at least
on $|x - x_0| < \rho$ or $|x-2| < 1$. $(1 < x < 3)$

Ex. Determine a lower bound for the radius of convergence of

① $(x^2-2x-3)y'' + xy' + 4y = 0$ about $x_0 = 4$.

sol: $y'' + \overbrace{\frac{x}{x^2-2x-3}}^{p} y' + \overbrace{\frac{4}{x^2-2x-3}}^{q} y = 0$

singular pts are $x^2-2x-3=0$
$(x-3)(x+1) = 0 \Rightarrow x=3,\ x=-1$

$\rho_1 = \text{dist.}(4,3) = 1$, $\rho_2 = \text{dist.}(4,-1) = 5$

$\rho = \min\{\rho_1, \rho_2\} = 1$

$\Rightarrow$ the series conv. for at least on
$|x-4| < 1$ or $3 < x < 5$.

② $y'' + \cancel{(\sin x)}$

(2) $(x^2-4x+5)y'' + xy' + y = 0$ about $x = 2$

Sol. Singular points : $x^2-4x+5=0$

$$x = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\rho_1 = \text{dist}(2+i, 2) = \text{dist}\left((2,1),(2,0)\right) = 1$$

$$\rho_2 = \text{dist}(2-i, 2) = \text{dist}\left((2,-1),(2,0)\right) = \sqrt{(2-2)^2+(-1-0)^2} = 1$$

$\therefore \rho = \min\{\rho_1, \rho_2\} = 1$

$\Rightarrow$ The series conv. for at least on $|x-2|<1$

or $-1 < x < 3$

(3) $y'' + (\sin x)y' + (1+x^2)y = 0$ about $x_0 = 0$

Sol. $p(x) = \sin x$ is analytic on $(-\infty, \infty)$, $\rho_1 = +\infty$

$q(x) = 1+x^2$ " " " " , $\rho_2 = +\infty$

$\Rightarrow$ ρ is infinite or $\rho = \infty$

(4) $y'' - e^x y + (1+x^2)y = 0$ about $x_0 = 2$

Ans. $\rho = \infty$.

5.4 Euler Equation, Regular singular points

Df: Consider the DE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \dots \quad (1)$$

where P, Q, & R are polynomials. Let x_0 be a singular point (i.e. $P(x_0) = 0$) and at least one of Q and R is not zero at x_0. Then

- $x = x_0$ is called regular singular point of Eq(1) if $\lim_{x \to x_0} (x - x_0)\dfrac{Q(x)}{P(x)}$ and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \text{ are finite.}$$

- If P, Q, & R are not polynomials in Eq(1) then we say that $x = x_0$ is regular singular point of Eq(1) if

$(x - x_0)\dfrac{Q(x)}{P(x)}$ and $(x - x_0)^2 \dfrac{R(x)}{P(x)}$ are analytic at $x = x_0$.

(i.e., have convergent Taylor series about $x = x_0$).

- A singular point of Eq (1) that is not regular singular point is called an <u>irregular singular point</u> of Eq (1).

<u>Ex.</u> Determine the singular points of the given d.e's. Determine whether they are regular or irregular.

① $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$.

<u>Sol:</u> The singular pts are where $2x(x-2)^2 = 0$

$\Rightarrow x = 0, \quad x = 2$.

$\boxed{x=0}$ $\quad (x-x_0)\dfrac{Q(x)}{P(x)} = (x-0)\left(\dfrac{3x}{2x(x-2)^2}\right) = \dfrac{3x^2}{2x(x-2)^2}$

- $\lim\limits_{x\to 0} \dfrac{3x^2}{2x(x-2)^2} = \lim\limits_{x\to 0} \dfrac{3x}{2(x-2)^2} = 0$ finite

- $\lim\limits_{x\to 0} (x-x_0)^2 \dfrac{R(x)}{P(x)} = \lim\limits_{x\to 0} x^{\cancel{2}} \cdot \dfrac{\cancel{x-2}}{2\cancel{x}(x-2)^{\cancel{2}}}$

$$= \lim_{x\to 0} \frac{x}{2(x-2)} = 0 \text{ finite}$$

$\Rightarrow$ $x=0$ is regular singular point.

$\boxed{x=2}$ $$\lim_{x\to 2}(x-2)\frac{Q(x)}{P(x)} = \lim_{x\to 2}\cancel{(x-2)}\frac{3\cancel{x}}{2\cancel{x}(x-2)^{\cancel{2}}}$$

$$= \frac{3}{2}\lim_{x\to 2}\frac{1}{x-2}$$

infinite.

$\therefore$ $x=2$ is irregular singular point.

(2) $x(3-x)y'' + (x+1)y' - 2y = 0$

sing pts are $x=0$, $x=3$.

$$\lim_{x\to 0}(x-0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x \cdot \frac{(x+1)}{x(3-x)} = \frac{1}{3} \text{ fini[te]}$$

$$\lim_{x\to 0}(x-0)^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2 \cdot \frac{-2}{x(3-x)}$$

$$= \lim_{x\to 0}\frac{-2x}{3-x} = \frac{0}{3} = 0$$

finite

$x=0$ is regular singular point.

$\boxed{x=3}$ $\lim\limits_{x\to 3}(x-3)\dfrac{x+1}{x(3-x)} = \lim\limits_{x\to 3}\dfrac{-(x+1)}{x}$

$= -\frac{4}{3}$ finite.

$\lim\limits_{x\to 3}(x-3)^2\cdot\dfrac{-2}{x(3-x)} = \lim\limits_{x\to 3}\dfrac{+2(x-3)}{x}$

$= \frac{0}{3} = 0$ finite

$\Rightarrow x=3$ is regular singular point.

(3) $x^2y'' - (3\sin x)y' + (1+x^2)y = 0$.

the singular point is $x=0$.

$(x-0)\dfrac{Q(x)}{P(x)} = x\cdot\left(\dfrac{-3\sin x}{x^2}\right)$

$= \dfrac{-3\sin x}{x}$

$= \dfrac{-3}{x}\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\right)$

$= -3 + \dfrac{3x^2}{3!} - \dfrac{3x^4}{5!} + \cdots$

is analytic at $x=0$.

$(x-0)^2\dfrac{R(x)}{P(x)} = x^2\cdot\dfrac{1+x^2}{x^2} = 1+x^2$ is analytic at $x=0$

$\Rightarrow x=0$ is regular singular pt.

H.W

(4) $x(1-x^2)^3 y'' + (1-x^2)^2 y' + 2(1+x)y = 0$

H.W

(5) $(x-\frac{\pi}{2})^2 y'' + (\cos x)y' + (\sin x)y = 0$.

Cauchy-Euler Equation.

A d.e of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

where $a_0, a_1, \ldots, a_n$ are constants is known as a Cauchy-Euler Eq. of nth order.

In this section we consider the homog. 2nd order eq. That is,

$$\boxed{a x^2 y'' + b x y' + c y = 0} \qquad (*)$$

Let $y = x^m$ be a sol. of $(*)$.

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

Substitute: $ax^2 \cdot m(m-1)x^{m-2} + bx \cdot m x^{m-1} + cx^m = 0$

$\Rightarrow am(m-1)x^m + bm x^m + cx^m = 0$

$\Rightarrow \left(am(m-1) + bm + c\right) x^m = 0.$

The aux. eq. is $\boxed{am^2 + (b-a)m + c = 0}$

we have three cases for the roots:

(i) If $m_1 \neq m_2 \in \mathbb{R}$, then

$$y_h = c_1 |x|^{m_1} + c_2 |x|^{m_2}$$

(ii) If $m_1 = m_2 = m \in \mathbb{R}$ (repeated reals).

$$y_h = c_1 |x|^m + c_2 |x|^m \ln|x|.$$

(iii) If $m = \alpha \pm \beta i$ (complex)

$$y_h = c_1 |x|^\alpha \cos(\beta \ln|x|) + c_2 |x|^\alpha \sin(\beta \ln|x|)$$

Ex. Solve the following d.e's

(1) $x^2 y'' - 4xy' + 6y = 0, \quad x > 0$

The aux. eq. is $m^2+(-4-1)m+6=0$

$$m^2-5m+6=0$$

$$(m-3)(m-2)=0$$

$$\Rightarrow m_1=3, \quad m_2=2$$

$$\therefore y_h = c_1 x^3 + c_2 x^2, \quad x>0.$$

(2) $4x^2y''+8xy'+y=0, \quad x>0.$

The aux. eq. is $4m^2+(8-4)m+1=0$

$$\Rightarrow (2m+1)^2=0$$

$$\Rightarrow m_1=m_2=\frac{-1}{2}.$$

$$y_h = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x.$$

(3) $4x^2y''+17y=0, \quad x>0.$

the aux. eq. is $4m^2-4m+17=0$

$$m = \frac{4 \pm \sqrt{16-4(4)(17)}}{2(4)}$$

$$= \frac{4 \pm 16i}{8} = \frac{1}{2} \pm 2i$$

$$y_h = c_1 x^{\frac{1}{2}} \cos(2\ln x) + c_2 x^{\frac{1}{2}} \sin(2\ln x),$$

$x > 0$.

(4) $xy'' - \frac{2}{x} y = x$, $x > 0$

Sol. multiply by x: $x^2 y'' - 2y = x^2$

this is a nonhomog. Euler Eq.

- y_h: $x^2 y'' - 2y = 0$

The aux. eq. is $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m_1 = 2$, $m_2 = -1$

$$y_h = c_1 x^2 + c_2 x^{-1}.$$

- y_p (use variation of parameters)

$\boxed{y_1 = x^2}$, $\boxed{y_2 = x^{-1}}$, $\boxed{g(x) = 1}$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0$$

$$v_1 = -\int \frac{y_2 g}{W} dx = -\int \frac{x^{-1} \cdot 1}{-3} = \frac{1}{3} \ln x.$$

$$V_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2 \cdot 1}{-3}\,dx = -\frac{1}{9}x^3.$$

$$y_p = V_1 y_1 + V_2 y_2 = \frac{1}{3}x^2 \ln x - \frac{1}{9}x^3 \cdot x^{-1}$$
$$= \frac{1}{3}x^2\left(\ln x - \frac{1}{3}\right)$$

absorb

$$\therefore y_g = y_h + y_p$$
$$= c_1 x^2 + c_2 x^{-1} + \frac{1}{3}x^2 \ln x - \frac{1}{9}x^2$$
$$= Ax^2 + Bx^{-1} + \frac{1}{3}x^2 \ln x.$$

H.W

(5) $x^2 y'' - x y' - 3y = \ln x, \quad x > 0.$

(6) $x y'' + \dfrac{y}{x} = \dfrac{\tan^{-1}(\ln x)}{x}, \quad x > 0.$

5.5 Series Solutions Near a regular Singular point, part I.

Our Aim. We need to solve the general second order lin eq. $P(x)y'' + Q(x)y' + R(x)y = 0$ in the neighborhood of a regular singular point $x = x_0$ as follows

ex. Find the first three nonzero terms of the series solution of the eq.

(1) $\boxed{2x^2y'' - xy' + (1+x)y = 0}$ which corresponds to the larger indicial root of the D.E around $x=0$.

Sol. $y'' - \frac{1}{2x}y' + \left(\frac{1+x}{2x^2}\right)y = 0$

step1

$$p(x) = (x-0)\left(\frac{1}{-2x}\right) = -\frac{1}{2}$$
$$q(x) = (x-0)^2\left(\frac{1+x}{2x^2}\right) = \frac{1}{2} + \frac{1}{2}x$$

analytic at $x=0$

$\Rightarrow$ $x=0$ is regular sing. pt.

step2 Indicial Equation $r(r-1) + a_0 r + b_0 = 0$

where a_0 = constant term in p.

b_0 = " " " q.

$\Rightarrow a_0 = -\frac{1}{2}, \quad b_0 = \frac{1}{2}$.

$\therefore$ Indicial eq. $r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$

$$r^2 - r - \frac{1}{2}r + \frac{1}{2} = 0$$

$$\Rightarrow r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \Rightarrow \boxed{2r^2 - 3r + 1 = 0}$$

Step 3 Indicial roots

$$2r^2-3r+1=0$$

$$\Rightarrow (2r-1)(r-1)=0 \Rightarrow \boxed{r_1=1} \; \boxed{r_2=\tfrac{1}{2}}$$

Step 4 For $\boxed{r_1=1}$ let $y=\sum_{n=0}^{\infty} a_n x^{n+1}$

$$y'=\sum_{n=0}^{\infty}(n+1)a_n x^n, \quad y''=\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

Substitute y, y', y'' into Eq (1),

$$2x^2\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} - \cancel{3}\sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\Rightarrow x^1\left[2\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty}(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}\right] = 0$$

$$\Rightarrow x\left[\sum_{n=0}^{\infty}\left(2n(n+1)a_n - (n+1)a_n + a_n\right)x^n + \sum_{n=1}^{\infty} a_{n-1}x^n\right]=0$$

$$\Rightarrow x\left[(0-\cancel{a_0}+\cancel{a_0}) + \sum_{n=1}^{\infty}\left(2n(n+1)a_n - (n+1)a_n + a_n + a_{n-1}\right)x^n\right]=0$$

$$\Rightarrow 2n(n+1)a_n - (n+1)a_n + a_n + a_{n-1} = 0, \quad n=1,2,\ldots$$

or

$$(2n^2+2n-n-\cancel{1}+\cancel{1})a_n = -a_{n-1}$$

$$n(2n+1)a_n = -a_{n-1}$$

$$\Rightarrow a_n = \frac{-1}{n(2n+1)}a_{n-1}, \quad n=1,2,\ldots$$

thus,

$$a_1 = \frac{-a_0}{3\cdot 1}, \quad a_2 = \frac{-a_1}{5\cdot 2} = \frac{a_0}{(3\cdot 5)(1\cdot 2)}$$

$$a_3 = -\frac{a_2}{7\cdot 3} = -\frac{a_0}{(3\cdot 5\cdot 7)(1\cdot 2\cdot 3)} \; \ldots$$

In general, $a_n = \dfrac{(-1)^n}{[3\cdot5\cdot7\cdot\cdots(2n+1)]\,n!}a_0, \quad n\geq 4.$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= x\left[a_0 + a_1x + a_2x^2 + \cdots\right.$$

$$= x\left[a_0 - \frac{a_0}{3\cdot1}x + \frac{a_0}{1\cdot2\cdot3\cdot5}x^2 + \cdots\right.$$

$$= a_0\Big[\underbrace{x - \frac{x^2}{3} + \frac{x^2}{30} + \cdots}_{y_1}$$

<u>steps</u> For $\boxed{r_2 = \frac{1}{2}}$

let $y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$, $y' = \sum_{n=0}^{\infty}(n+\frac{1}{2})a_n x^{n-\frac{1}{2}}$

$$y'' = \sum_{n=0}^{\infty}(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n x^{n-\frac{3}{2}}.$$

<u>Subst.</u>

$$2x^2\sum_{n=0}^{\infty}(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n x^{n-\frac{3}{2}} - \sum_{n=0}^{\infty}(n+\tfrac{1}{2})a_n x^{n+\frac{1}{2}}$$
$$+\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty}a_n x^{n+\frac{3}{2}} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}(2n+1)(n-\tfrac{1}{2})a_n x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty}(n+\tfrac{1}{2})a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty}a_n x^{n+\frac{3}{2}} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(2n+1)(n-\tfrac{1}{2}) - (n+\tfrac{1}{2}) + 1\right]a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty}a_n x^{n+\frac{3}{2}} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(2n^2-n+\cancel{n}-\cancel{\tfrac{1}{2}}-\cancel{n}-\cancel{\tfrac{1}{2}}+\cancel{1}\right)a_n x^{n+\frac{1}{2}}+\sum_{n=0}^{\infty}a_n x^{n+\frac{3}{2}}=0.$$

$$x^{\frac{1}{2}}\left[\sum_{n=0}^{\infty}(2n^2-n)a_n x^n+\sum_{n=0}^{\infty}a_n x^{n+1}\right]=0.$$

$$\sum_{n=0}^{\infty}(2n^2-n)a_n x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0,$$

$$0.a_0x^0+\sum_{n=1}^{\infty}\left[n(2n-1)a_n+a_{n-1}\right]x^n=0,$$

$$\Rightarrow \quad a_n=-\frac{a_{n-1}}{n(2n-1)},\quad n=1,2,\ldots$$

$$a_1=-\frac{a_0}{1\cdot 1}=-a_0.$$

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$$a_2=-\frac{a_1}{2\cdot 3}=\frac{a_0}{(1\cdot 2)(1\cdot 3)}=\frac{a_0}{6}.$$

$$a_3=-\frac{a_2}{3\cdot 5}=-\frac{a_0}{(1\cdot 2\cdot 3)(1\cdot 3\cdot 5)}=-\frac{a_0}{90}$$

$\vdots$

$$y=\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}}$$

$$=x^{\frac{1}{2}}\left[a_0+a_1x+a_2x^2+\cdots\right.$$

$$=x^{\frac{1}{2}}\left[a_0-a_0x+\frac{a_0}{6}x^2-\frac{a_0}{90}x^3+\cdots\right.$$

$$=a_0\, x^{\frac{1}{2}}\left[1-x+\frac{1}{6}x^2-\frac{1}{90}x^3+\cdots\right]$$

y_2

Ex. (H.W) Find the first three nonzero terms of the series solution of the eq.

$$4x\,y'' + 2y' + y = 0 \quad \text{about } x=0.$$

which corresponds to the larger indicial root of the D.E.

Ans: $\boxed{r_1 = 0}$, $\boxed{r_2 = \frac{1}{2}}$

For $\boxed{r_1 = 0}$ $\Rightarrow$ the recurrence relation is

$$a_n = \frac{-1}{2n(2n-1)}\,a_{n-1}, \quad n = 1, 2, \ldots$$

For $\boxed{r_2 = \frac{1}{2}}$ $\Rightarrow$ the recurrence relation is

$$a_n = \frac{-1}{2n(2n+1)}\,a_{n-1}, \quad n = 1, 2, \ldots$$

$\vdots$

CH6 The Laplace Transform

6.1 Definition of the Laplace transform

Review [Calculus II]. Improper Integrals

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt \text{, where } A>0 \text{ real.}$$

If $\int_a^A f(t)\,dt$ exists for $A>a$, and the limits as $A\to\infty$ exists, then the improper integral is said to be converge. Otherwise the integral is said to be diverge.

ex. $$\int_0^\infty e^{\alpha t}\,dt = \lim_{A\to\infty} \int_0^A e^{\alpha t}\,dt$$
$$= \lim_{A\to\infty} \left.\frac{e^{\alpha t}}{\alpha}\right|_0^A \text{, } \alpha \neq 0$$
$$= \lim_{A\to\infty} \frac{e^{\alpha A}-1}{\alpha}$$
$$= \begin{cases} -\frac{1}{\alpha}, & \text{if } \alpha<0 \\ \text{div.}, & \text{if } \alpha>0 \end{cases}$$

If $\alpha=0$, $\int_0^\infty e^{\alpha t}\,dt = \int_0^\infty 1\,dt = \infty$ div.

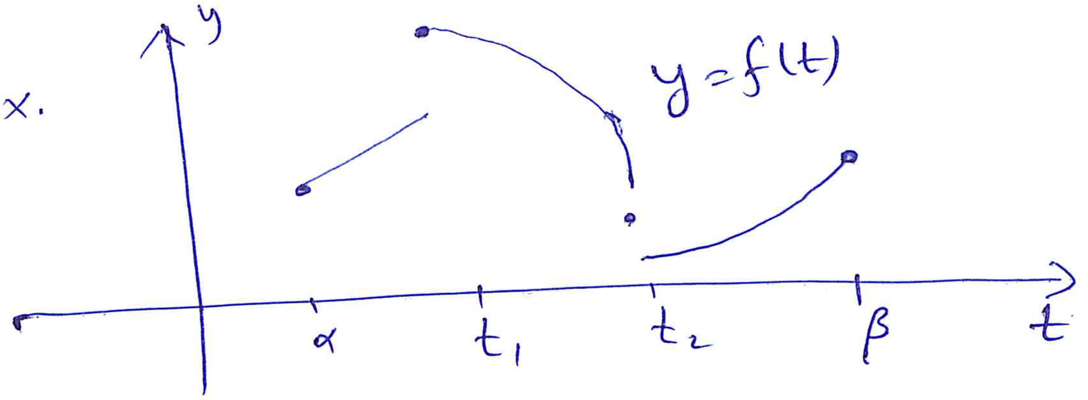
ex. $\int_1^\infty \frac{1}{t}\,dt = \lim_{A\to\infty}\int_1^A \frac{1}{t}\,dt$

$= \lim_{A\to\infty} \ln|t|\Big|_1^A$

$= \lim_{A\to\infty} (\ln A - \ln 1) = \infty$ div.

Df: A function f is said to be piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities.

ex.



A piecewise continuous function $y = f(t)$

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{\beta} f(t)\,dt.$$

Rmk: If f is a piecewise continuous on $a \leq t \leq b$, then $\int_a^b f(t)\,dt$ exists. However, piecewise continuity is not enough to ensure convergence of $\int_a^\infty f(t)\,dt$. In this case, we use

Comparison test.

The Laplace Transform

<u>Df.</u> An integral transform is a relation of the form

$$\boxed{F(s) = \int_{\alpha}^{\beta} f(t)\, K(s,t)\, dt} \quad (*)$$

where $K(s,t)$ is called the <u>kernel</u> of the transformation and α, β are also given. It is possible that $\alpha = -\infty$ or $\beta = \infty$ or both. The relation $(*)$ transforms f into another function F, which is called the transform of f.

Df. (Laplace Transform) ($\mathcal{L}.T$)

The Laplace transform of f is defined as

$$\boxed{\mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)\, e^{-st}\, dt = F(s)} \quad (**)$$

provided this improper integral converges.

<u>Thm</u> (Existence of $\mathcal{L}.T$).

Suppose that ① f is piecewise continuous

(1-7)

on $0 \le t \le A$, for any $A > 0$.

② $|f(t)| \le K e^{at}$, $t \ge M$, where K, a, and M are real constants, K and M are positive. Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by (*) exists for $s > a$.

Examples.

① $$\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$
$$= \lim_{A\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^A$$
$$= \lim_{A\to\infty} \frac{1 - e^{-sA}}{s} = \frac{1}{s},\ s > 0$$

In general, $\boxed{\mathcal{L}\{k\} = \frac{k}{s},\ s > 0}$ where k is constant.

ex. $\mathcal{L}\{2020\} = \frac{2020}{s}$, $s > 0$.

② $\mathcal{L}\{e^{kt}\} = ??$, where k is constant.

$$= \int_0^\infty e^{kt} e^{-st}\, dt = \int_0^\infty e^{-(s-k)t}\, dt$$

$$= \lim_{A\to\infty} \left. \frac{e^{-(s-k)t}}{-(s-k)} \right|_0^A$$

$$= \lim_{A\to\infty} \frac{1-e^{-(s-k)A}}{s-k} = \frac{1}{s-k}, \text{ if } s>k.$$

$$\therefore \boxed{\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, \quad s>k}$$

ex. $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad s>2.$

③ $\mathcal{L}\{t^2\} = ??$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$$

$$= \lim_{A\to\infty} \int_0^A t^2 e^{-st}\,dt$$

$$\begin{array}{ccc} t^2 & & e^{-st} \\ & \xrightarrow{(+)} & \frac{e^{-st}}{-s} \\ 2t & & \\ & \xrightarrow{(-)} & \frac{e^{-st}}{s^2} \\ 2 & & \\ & \xrightarrow{(+)} & -\frac{e^{-st}}{s^3} \\ 0 & & \end{array}$$

$$= \lim_{A\to\infty} \left. \left(-\frac{t^2 e^{-st}}{s} - \frac{2t\,e^{-st}}{s^2} - 2\frac{e^{-st}}{s^3} \right) \right|_0^A$$

$$= \lim_{A\to\infty} \left[\frac{-\frac{A^2}{s} - \frac{2A}{s^2} - \frac{2}{s^3}}{e^{sA}} + \frac{2}{s^3} \right]$$

L'Hopital

$$= 0 + \frac{2}{s^3} = \frac{2!}{s^3}, \quad s>0$$

In general, $\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$, $s>0$

ex. $\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$.

ex. $\mathcal{L}\{t^7\} = \frac{7!}{s^8}$.

(4) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$, $s>0$.

(5) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$, $s>0$

(6) $\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$, $s>|k|$.

(7) $\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$, $s>|k|$.

(8) $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$, $s>a$.

(9) $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$, $s>a$.

(10) $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$, $n=0,1,2,\ldots$
$s>a$

(11) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}\right)$

(12) (Linearity)

$$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$$

Pf. $\mathcal{L}\{\alpha f(t) \pm \beta g(t)\}$

$$= \int_0^\infty (\alpha f(t) \pm \beta g(t))\, e^{-st}\, dt$$

$$= \alpha \int_0^\infty f(t)\, e^{-st}\, dt \pm \beta \int_0^\infty g(t)\, e^{-st}\, dt$$

$$= \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\} \quad \blacksquare$$

Examples

(1) $\mathcal{L}\{4e^{-2t} - 3\sin 4t\}$

$$= 4\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\}$$

$$= 4\,\frac{1}{s+2} - 3 \cdot \frac{4}{s^2+(4)^2} = \frac{4}{s+2} - \frac{12}{s^2+16}$$

(2) $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$

$$= \mathcal{L}\{\tfrac{1}{2}\} - \tfrac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2} \cdot \frac{s}{s^2+4}$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

(3) $\mathcal{L}\{t^2 e^t\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\mathcal{L}\{e^t\}\right)$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

let $F(s) = \dfrac{1}{s-1} \Rightarrow F'(s) = \dfrac{-1}{(s-1)^2} = -(s-1)^{-2}$

$$F''(s) = 2(s-1)^{-3}$$

$\therefore \mathcal{L}\{t^2 e^t\} = F''(s) = \dfrac{2}{(s-1)^3}.$

(4) $\mathcal{L}\{\cosh 6t\} = \dfrac{s}{s^2-36}, \ s > 6.$

(5) $\mathcal{L}\{e^{2t}\sin 6t\} = \dfrac{6}{(s-2)^2+36}$ $\left(\begin{matrix} a=2 \\ b=6 \end{matrix} \text{ in } (8)\right).$

(6) $\mathcal{L}\{\sin(2t+\frac{\pi}{3})\}$

$= \mathcal{L}\{\sin 2t \cos\frac{\pi}{3} + \cos 2t \sin\frac{\pi}{3}\}$

$= \frac{1}{2}\mathcal{L}\{\sin 2t\} + \frac{\sqrt{3}}{2}\mathcal{L}\{\cos 2t\}$

$= \frac{1}{2}\cdot\dfrac{2}{s^2+4} + \frac{\sqrt{3}}{2}\cdot\dfrac{s}{s^2+4}$

$= \dfrac{1+\frac{\sqrt{3}}{2}s}{s^2+4}.$

ex. Prove that $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}, s>|k|$

Pf. $\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\dfrac{e^{kt}+e^{-kt}}{2}\right\}$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right], s>|k|$$

$$= \frac{1}{2}\left[\frac{s+\cancel{k}+s-\cancel{k}}{(s-k)(s+k)}\right]$$

$$= \frac{s}{s^2-k^2}, \quad s>|k| \quad \blacksquare$$

Similarly, $\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\dfrac{e^{kt}-e^{-kt}}{2}\right\}$

$$= \frac{1}{2}\left(\mathcal{L}\{e^{kt}\} - \mathcal{L}\{e^{-kt}\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s}+k-\cancel{s}+k}{(s-k)(s+k)}\right)$$

$$= \frac{k}{s^2-k^2}, \quad s>|k| \quad \blacksquare$$

6.2 Solutions of Initial value problem

In this section we show how the Laplace transform can be used to solve IVP's for linear DE with constant coefficients.

First, we need the following.

The inverse Laplace transform

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Ex. Find the inverse $\mathcal{L}.T$.

① $\mathcal{L}^{-1}\left\{\frac{2020}{s}\right\} = 2020.$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$

$= \frac{1}{2}\cdot t^2$

③ $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+4}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= 2\cos 2t - \frac{3}{2}\sin 2t.$

④ $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^{3+1}}\right\}$

$= \frac{1}{3!}e^{2t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$

$= \frac{1}{6}e^{2t}\cdot t^3.$

⑤ $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+(3)^2}\right\}$

$= e^{2t}\cos 3t + \frac{2}{3}e^{2t}\sin 3t.$

⑥ $\mathcal{L}^{-1}\left\{\frac{2s+2}{s^2+2s+6}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+5}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+5}\right\}$

$= 2e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\}$

$= 2e^{-t}\cos(\sqrt{5}\,t).$

7. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)(s+1)}\right\}$

Now, $\dfrac{1}{(s-3)(s+1)} = \dfrac{A}{s-3} + \dfrac{B}{s+1}$

$\Rightarrow \boxed{1 = A(s+1) + B(s-3)}$

$\boxed{s=-1}$ $\quad 1 = B(-4) \Rightarrow \boxed{B = -1/4}$

$\boxed{s=3}$ $\quad 1 = 4A \Rightarrow \boxed{A = 1/4}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s-3} - \frac{\frac{1}{4}}{s+1}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}.$$

8. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+2)(s^2+4)}\right\}$

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

$$s = A(s^2+4) + (Bs+C)(s+2)$$

$$s = As^2 + 4A + Bs^2 + 2Bs + Cs + 2C$$

$$\boxed{s = (A+B)s^2 + (2B+C)s + (4A+2C)}$$

$\Rightarrow \boxed{A+B=0}, \quad \boxed{2B+C=1}, \quad \boxed{4A+2C=0}$

Now, $-2(2B+C=1 \Rightarrow -4B-2C=-2$

$$\begin{aligned} 4A+2C&=0 \\ \hline 4A-4B&=-2 \\ \boxed{A-B=-\tfrac{1}{2}} \end{aligned}$$

Next, solve

$$\begin{aligned} A+B&=0 \\ A-B&=-\tfrac{1}{2} \\ \hline 2A&=-\tfrac{1}{2} \end{aligned} \Rightarrow \boxed{A=-1/4} \Rightarrow \boxed{B=\tfrac{1}{4}}$$

$$\boxed{C=-2A=\tfrac{1}{2}}.$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s+2}+\frac{\frac{1}{4}s+\frac{1}{2}}{s^2+4}\right\}$$

$$= -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\frac{1}{4}e^{-2t}+\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t.$$

Laplace of derivatives

1) $\mathcal{L}\{y(t)\} = Y(s).$

2) $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

3) $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

④ $\mathcal{L}\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$.

⋮

In general, $\mathcal{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots$

$\cdots - y^{(n-1)}(0)$.

Now, we show how the Laplace transform can be used to solve IVP's.

Ex. Use the Laplace transform to solve the IVP.
$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

Sol. Take L.T for both sides:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$[s^2 Y - s y(0) - y'(0)] - [sY - y(0)] - 2Y = 0$$

$$s^2 Y - s - sY + 1 - 2Y = 0$$

$$(s^2 - s - 2) Y = s - 1$$

$$\Rightarrow Y = \frac{s-1}{s^2 - s - 2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s+1)}\right\}$$

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\Rightarrow \boxed{s-1 = A(s+1) + B(s-2)}$$

$$\boxed{s=-1} \Rightarrow -2 = -3B \Rightarrow \boxed{B = \tfrac{2}{3}}$$

$$\boxed{s=2} \Rightarrow 1 = 3A \Rightarrow \boxed{A = 1/3}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Ex. Use the $\mathcal{L}.T$ to solve the IVP

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2, \; y'(0) = -1 \end{cases}$$

Sol: Take $\mathcal{L}.T$ for both sides:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$$

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + Y = 4 \cdot \frac{1}{s+1}$$

$$s^2Y - 2s + 1 + 2sY - 4 + Y = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)Y = 2s + 3 + \frac{4}{s+1}$$

$$Y(s) = \frac{(2s+3)(s+1)+4}{(s+1)(s+1)^2} = \frac{2s^2+5s+7}{(s+1)^3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s^2+5s+7}{(s+1)^3}\right\}$$

$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\Rightarrow 2s^2+5s+7 = A(s+1)^2 + B(s+1) + C$$

$$2s^2+5s+7 = A(s^2+2s+1) + Bs + B + C$$

$$\boxed{2s^2+5s+7 = As^2 + (2A+B)s + (A+B+C)}$$

s^2: $\boxed{A = 2}$

s: $2A+B=5 \Rightarrow B = 5-2A = 5-4 = 1 \Rightarrow \boxed{B=1}$

s^0: $A+B+C=7 \Rightarrow 2+1+C=7$ $\boxed{C=4}$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1!}{(s+1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2!}{(s+1)^3}\right\}$$

$$= 2e^{-t} + e^{-t}.t + 2.e^{-t}.t^2$$

$$= (2+t+2t^2)e^{-t}$$

Ex. Solve using $\mathcal{L}\cdot T$

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = y''(0) = y'''(0) = 0, \quad y'(0) = 1 \end{cases}$$

Sol. Take $\mathcal{L}\cdot T$:

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^4 Y - \cancel{s^3 y(0)}^{\,0} - s^2 y'(0) - \cancel{s y''(0)}^{\,0} - \cancel{y'''(0)}^{\,0} - Y = 0$$

$$(s^4 - 1) Y = s^2 \Rightarrow Y = \frac{s^2}{s^4 - 1}$$

$$\therefore \; y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s^2}{(s-1)(s+1)(s^2+1)}\right\}$$

Now, $$\frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs + D}{s^2+1}$$

$$\Rightarrow \boxed{s^2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1)}$$

$\boxed{s=1} \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

$\boxed{s=-1} \Rightarrow 1 = B(-2)(2) \Rightarrow \boxed{B = -\frac{1}{4}}$

$\boxed{s=0} \Rightarrow A - B - D = 0 \Rightarrow \frac{1}{4} + \frac{1}{4} - D = 0 \Rightarrow \boxed{D = \frac{1}{2}}$

$\boxed{s=2} \Rightarrow 4 = A(3)(5) + B(1)(5) + (2C + D)(3)$

$4 = \frac{15}{4} - \frac{5}{4} + 6C + \frac{3}{2} \Rightarrow \boxed{C = 0}$

$$\therefore y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t.$$

6.3 Step Functions

Thm (1st Translation Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$, then

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}_{s \to s-a} = F(s-a)$$

OR $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} \mathcal{L}^{-1}\{F(s)\}$

ex. ① $\mathcal{L}\{e^{5t} \cdot t^3\} = \mathcal{L}\{t^3\}_{s \to s-5} = \dfrac{3!}{(s-5)^4}$

② $\mathcal{L}\{e^{-2t} \cos 4t\} = \mathcal{L}\{\cos 4t\}_{s \to s+2}$

$$= \frac{s}{s^2+16} \Big|_{s \to s+2}$$

$$= \frac{s+2}{(s+2)^2+16}$$

③ $\mathcal{L}^{-1}\left\{ \dfrac{2s+5}{(s-3)^2} \right\}$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$\Rightarrow \boxed{2s+5 = A(s-3) + B}$

$\boxed{s=3}$: $\boxed{11 = B}$, $\boxed{s=0}$: $5 = -3A + 11$

$\Rightarrow \boxed{A = 2}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11\,e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 2e^{3t} + 11\,e^{3t}\cdot t.$$

$$= (2+11t)\,e^{3t}.$$

④ $\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{2}s+\frac{5}{3}}{s^2+4s+6}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}(s+2)+\frac{2}{3}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(\sqrt{2})^2}\right\} + \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}\right\}$$

$$= \frac{1}{2}e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} + \frac{2}{3\sqrt{2}}e^{-2t}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}t).$$

⑤ Solve the IVP

$$\begin{cases} y''-6y'+9y = t^2e^{3t} \\ y(0)=2,\ y'(0)=17 \end{cases}$$ by using $\mathcal{L}$.T.

Sol: Take. L-T :

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$s^2 Y - s y(0) - y'(0) - 6[sY - y(0)] + 9Y = \frac{2!}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y - 2s - 17 + 12 = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s-3)^2 Y = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$y = \mathcal{L}^{-1}\{Y\}$$

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}}_{\text{Ex } ③} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}$$

$$= (2 + 11t)e^{3t} + \frac{1}{12} e^{3t} \cdot t^4 .$$

$$= \left(2 + 11t + \frac{1}{12}t^4\right) e^{3t} .$$

H.W Solve by using L-T

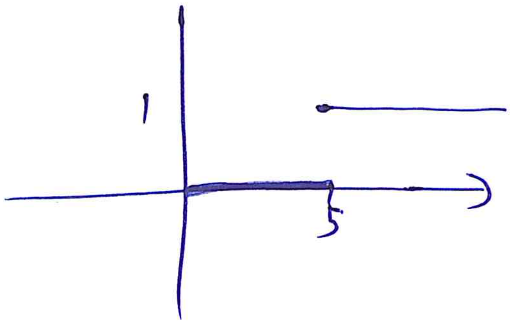
$$\begin{cases} y'' + 4y' + 6y = 1 + e^{-t} \\ y(0) = y'(0) = 0 \end{cases}$$

Df. (The unit step function or Heaviside functio

The unit step function or Heaviside function is defined by

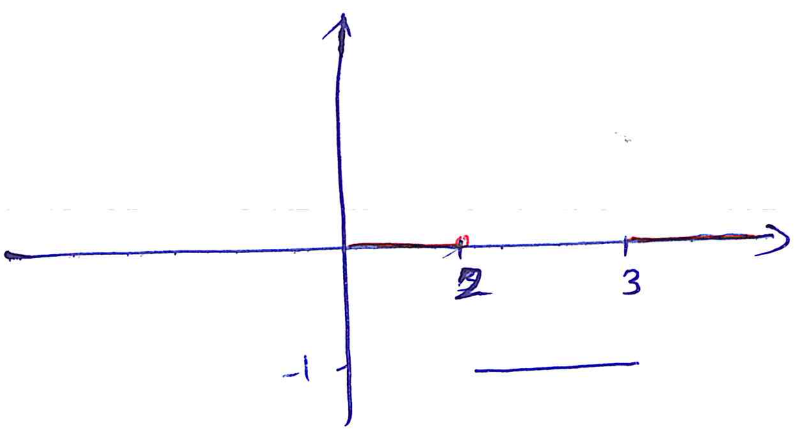
$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}, \quad c \geq 0$$

ex. $u_5(t) = \begin{cases} 0, & t < 5 \\ 1, & t \geq 5 \end{cases}$



ex. sketch the graph of $y = u_3(t) - u_2(t)$

$$y = \begin{cases} 0-0, & 0 \leq t < 2 \\ 0-1, & 2 \leq t < 3 \\ 1-1, & t \geq 3 \end{cases} = \begin{cases} 0, & 0 \leq t < 2 \text{ or } t \geq 3 \\ -1, & 2 \leq t < 3 . \end{cases}$$



Rmk. The unit step function can be used to write a piecewise function in a compact form as follows.

ex. $$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & a \le t < b \\ k(t), & t \ge b \end{cases}$$

Write f in a compact form

Sol. $$f(t) = g(t) + \big(h(t) - g(t)\big)\mathcal{U}_a(t) + \big(k(t) - h(t)\big)\mathcal{U}_b(t)$$

ex. Write in a compact form

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4 \\ \sin t + \cos(t - \frac{\pi}{4}), & t \ge \pi/4 . \end{cases}$$

Sol. $$f(t) = \sin t + \cos(t - \tfrac{\pi}{4})\,\mathcal{U}_{\frac{\pi}{4}}(t).$$

ex. Express $f(t)$ in terms of $\mathcal{U}_c(t)$ where

$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 2, & 5 \le t < 9 \\ -1, & t \ge 9 \end{cases}$$

Sol. $$f(t) = 20t + (2 - 20t)\,\mathcal{U}_5(t) - 3\,\mathcal{U}_9(t).$$

Ex. Find $\mathcal{L}\{u_c(t)\}$

Sol. $\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\, e^{-st}\, dt$

$$= \int_c^\infty 1 \cdot e^{-st}\, dt$$

$$= \lim_{A\to\infty} \int_c^A e^{-st}\, dt$$

$$= \lim_{A\to\infty} \left. \frac{e^{-st}}{-s} \right|_c^A$$

$$= \lim_{A\to\infty} \frac{e^{-sA} - e^{-sc}}{-s} = \frac{e^{-cs}}{s}.$$

$$\therefore \boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

ex. $\mathcal{L}\{u_2(t)\} = \dfrac{e^{-2s}}{s}$

ex. $\mathcal{L}^{-1}\left\{\dfrac{e^{-6s}}{s}\right\} = u_6(t).$

Thm (2nd Translation theorem)

If $\mathcal{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\,\mathcal{U}_a(t)\} = e^{-as} F(s), \quad s > a$$

OR $\mathcal{L}\{f(t)\,\mathcal{U}_a(t)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

and $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\,\mathcal{U}_a(t)$

$$= \mathcal{L}^{-1}\{F(s)\}_{t \to t-a} \cdot \mathcal{U}_a(t).$$

Ex. Find $\mathcal{L}\{t^2\,\mathcal{U}_3(t)\}$

$$= e^{-3s}\,\mathcal{L}\{t^2\}_{t \to t+3}$$

$$= e^{-3s}\,\mathcal{L}\{(t+3)^2\}$$

$$= e^{-3s}\,\mathcal{L}\{t^2 + 6t + 9\}$$

$$= e^{-3s}\left(\frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

Ex. Find $\mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} 0, & t < \pi \\ t-\pi, & \pi \le t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$$

Sol. 1st Write f in a compact form

$$f(t) = 0 + (t-\pi-0)\,\mathcal{U}_{\pi}(t) + (0-t+\pi)\,\mathcal{U}_{2\pi}(t)$$

$$= (t-\pi)\,\mathcal{U}_{\pi}(t) + (\pi-t)\,\mathcal{U}_{2\pi}(t)$$

$$\therefore\ \mathcal{L}\{f(t)\} = \mathcal{L}\{(t-\pi)\,\mathcal{U}_{\pi}(t)\} + \mathcal{L}\{(\pi-t)\,\mathcal{U}_{2\pi}(t)\}$$

$$= e^{-\pi s}\,\mathcal{L}\{t+\cancel{\pi}-\cancel{\pi}\} + e^{-2\pi s}\,\mathcal{L}\{\pi-(t+2\pi)\}$$

$$= e^{-\pi s}\,\mathcal{L}\{t\} + e^{-2\pi s}\,\mathcal{L}\{-t-\pi\}$$

$$= e^{-\pi s}\cdot\frac{1}{s^2} + e^{-2\pi s}\left(\frac{-1}{s^2} - \frac{\pi}{s}\right).$$

Ex. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+4}\right\}$

$$= \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2+4}\right\}$$

$$= \mathcal{U}_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}_{t\to t-2} = \frac{1}{2}\,\mathcal{U}_2(t)\,\sin 2(t-2)$$

ex. $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} = \mathcal{L}^{-1}\left\{e^{-4s}\cdot\frac{1}{s}\right\}$

$= \mathcal{U}_4(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}_{t\to t-4}$

$= \mathcal{U}_4(t)\cdot 1 = \mathcal{U}_4(t).$

In general, $\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\} = \mathcal{U}_c(t)}$

ex. $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2}\right\}$

$= t - \mathcal{U}_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{t\to t-2}$

$= t - \mathcal{U}_2(t)\cdot(t-2)$

ex. $\mathcal{L}^{-1}\left\{\frac{2(s-1)\,e^{-2s}}{s^2-2s+2}\right\}$

$= 2\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\,e^{-2s}\right\}$

$= 2\,\mathcal{U}_2(t)\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}_{t\to t-2}$

$= 2\,\mathcal{U}_2(t)\,\left(e^t\cos t\right)\Big|_{t\to t-2}$

$= 2\,\mathcal{U}_2(t)\,e^{t-2}\cos(t-2).$

6.4 Differential Equations with discontinuous Forcing functions

In this section, we solve some DEs in which the nonhomogeneous term or forcing function is discontinuous.

Ex ① Solve using Laplace transform

$$\begin{cases} y'' + 4y = \sin t \, \mathcal{U}_{2\pi}(t) \\ y(0) = 1, \; y'(0) = 0 . \end{cases}$$

Sol. Take L.T :

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t \, \mathcal{U}_{2\pi}(t)\}$$

$$s^2 Y - s y(0) - \cancel{y'(0)}^{\,0} + 4Y = e^{-2\pi s} \mathcal{L}\{\sin t\}_{t \to t+2\pi}$$

$$(s^2+4)Y - s = e^{-2\pi s} \mathcal{L}\{\sin(t+2\pi)\}$$

$$= e^{-2\pi s} \mathcal{L}\{\sin t\} = e^{-2\pi s} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow Y = \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}$$

$$\boxed{y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{U}_{2\pi}(t) \, \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}_{t \to t-}}_{(*)}}$$

Now, $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

s^3 terms: $A + C = 0$ — (1)

s^2 terms: $B + D = 0$ — (2)

s terms: $4A + C = 0$ — (3)

s^0 terms: $4B + D = 1$ — (4)

(3) − (1): $4A - A = 0 \Rightarrow \boxed{A = 0} \Rightarrow \boxed{C = 0}$

(4) − (2): $4B - B = 1 \Rightarrow \boxed{B = \frac{1}{3}} \Rightarrow \boxed{D = -\frac{1}{3}}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin(2t).$$

Back to (*):

$$y = \cos 2t + \mathcal{U}_{2\pi}(t)\left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2(t-2\pi))\right]$$

$$= \cos 2t + \mathcal{U}_{2\pi}(t)\left[\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right]$$

$$= \begin{cases} \cos 2t, & 0 \le t < 2\pi \\ \cos 2t + \frac{1}{3}\sin t - \frac{1}{6}\sin 2t, & t \geq 2\pi. \end{cases}$$

ex: ② Solve $\begin{cases} y'' + y = f(t) \\ y(0)=0,\ y'(0)=2 \end{cases}$, where

$$f(t) = \begin{cases} \frac{1}{2}t, & 0 \le t < 6 \\ 3, & t \ge 6. \end{cases}$$

Sol. $f(t) = \frac{1}{2}t + (3 - \frac{1}{2}t)\mathcal{U}_6(t)$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\mathcal{L}\{t\} + \mathcal{L}\{(3-\frac{1}{2}t)\mathcal{U}_6(t)\}$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} + e^{-6s}\,\mathcal{L}\{3-\frac{1}{2}t\}_{t\to t+6}$$

$$= \frac{1}{2s^2} + e^{-6s}\,\mathcal{L}\{3-\frac{1}{2}(t+6)\}$$

$$= \frac{1}{2s^2} + e^{-6s}\left(-\frac{1}{2}\cdot\frac{1}{s^2}\right)$$

$$= \frac{1-e^{-6s}}{2s^2}$$

Now,

Take $\mathcal{L}.T$:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$s^2Y - \cancel{sy(0)}^{\,0} - y'(0) + Y = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s}$$

$$(s^2+1)Y = 2 + \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s}$$

$$Y = \frac{2}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{1}{2s^2(s^2+1)}e^{-6s}$$

$$y = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2s^2(s^2+1)}\right\}$$

$$- \mathcal{U}_6(t)\, \mathcal{L}^{-1}\left\{\frac{1}{2s^2(s^2+1)}\right\}_{t \to t-6} \qquad (**)$$

Now,

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}\right]$$

$$\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

s^3: $A + C = 0$

s^2: $B + D = 0$

s: $A = 0 \Rightarrow C = 0$

s^0: $B = 1 \Rightarrow D = -1$

$$\therefore \frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{2s^2(s^2+1)}\right\} = \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(t - \sin t)$$

Back to $(**)$:

$$y = 2\sin t + \frac{1}{2}t - \frac{1}{2}\sin t$$
$$- \frac{1}{2}\mathcal{U}_6(t)\left[t-6-\sin(t-6)\right].$$
$$= \frac{3}{2}\sin t + \frac{1}{2}t - \frac{1}{2}\mathcal{U}_6(t)\left(t-6-\sin(t-6)\right).$$

Ex (3)(H.w) Solve using $\mathcal{L}.T$:

$$\begin{cases} 2y'' + y' + 2y = g(t) \\ y(0) = y'(0) = 0, \end{cases}$$ where

$$f(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ and } t \ge 20 \end{cases} = \mathcal{U}_5(t) - \mathcal{U}_{20}(t)$$

Ex (4) (H.w) Solve $\begin{cases} y'' + 4y' + 3y = h(t) \\ y(0) = y'(0) = 0, \end{cases}$

where $h(t) = \begin{cases} 0, & 0 \le t < 2 \\ -1, & t \ge 2. \end{cases}$

6.5 Impulse Functions

In some application it is necessary to deal with phenomena of an impulsive nature - for example voltages or forces of large magnitude that act over very short time intervals. Such problems often lead to d.e's of the form

$ay'' + by' + cy = g(t)$, where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ and is zero otherwise.

We define $I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$ or since $g(t) = 0$ outside $(t_0-\tau, t_0+\tau)$,

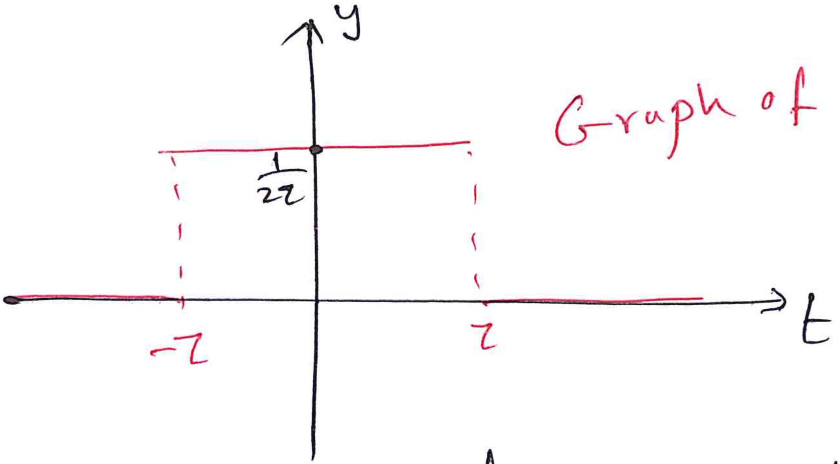
$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt.$$

In a mechanical system, where $g(t)$ is a force, $I(\tau)$ is the total impulse of the force $g(t)$ over the time interval $(t_0-\tau, t_0+\tau)$.

In particular, let us suppose that t_0 is zero and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & t \le -\tau \text{ or } t \ge \tau. \end{cases}$$

where τ is small positive constant.

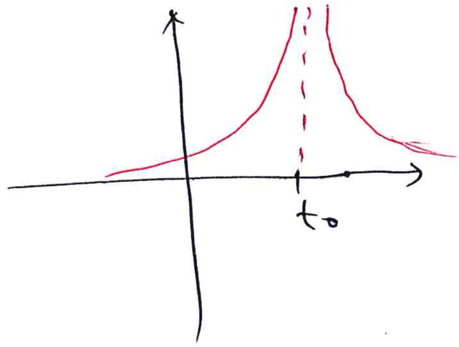


Notice that $\lim_{\tau \to 0} d_\tau(t) = 0, \quad t \ne 0$

Dirac Delta function

The Dirac delta function is defined as

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0. \end{cases}$$



Properties. The Dirac delta function satisfies the following properties:

① $\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1.$

② $\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\, dt = f(t_0)$.

Ex. $\int_{-\infty}^{\infty} \delta(t-\frac{\pi}{2}) \sin t\, dt = \sin\frac{\pi}{2} = 1$.

ex. $\int_{-\infty}^{\infty} 2\delta(t-\frac{\pi}{3}) \cos t\, dt = 2\cos\frac{\pi}{3} = 2(\frac{1}{2})$
$= 1$.

③ $\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$, $t_0 > 0$.

④ $\mathcal{L}\{\delta(t-t_0)\, f(t)\} = e^{-t_0 s} f(t_0)$.

ex. $\mathcal{L}\{\delta(t-6)\} = e^{-6s}$.

ex. $\mathcal{L}\{\delta(t-\frac{\pi}{4}) \sin t\} = e^{-\frac{\pi}{4}s} \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}s}$.

ex. $\mathcal{L}^{-1}\{1\} = \mathcal{L}^{-1}\{e^{-0.s}\} = \delta(t-0)$
$= \delta(t)$.

Ex. Solve the DE

$$\begin{cases} y'' + y = 4\delta(t-2\pi) \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

(208)

Sol: Take L.T:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\,\mathcal{L}\{\delta(t-2\pi)\}$$

$$s^2Y - sy(0) - y'(0) + Y = 4e^{-2\pi s}.$$

$$(s^2+1)Y = s + 4e^{-2\pi s}.$$

$$Y = \frac{s}{s^2+1} + \frac{4}{s^2+1}e^{-2\pi s}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-2\pi s}\right\}$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t \to t-2\pi}$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\sin(t-2\pi)$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\sin t$$

$$= \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$$

(209)

Ex. Solve the IVP

$$\begin{cases} 2y'' + y' + 4y = 2\,\delta(t-\frac{\pi}{6})\sin t \\ y(0) = y'(0) = 0 \end{cases}$$

Sol. Take L.T:

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-\tfrac{\pi}{6})\sin t\}$$

$$2[s^2Y - \cancel{sy(0)}^{\,0} - \cancel{y'(0)}^{\,0}] + sY - \cancel{y(0)}^{\,0} + 4Y = 2e^{-\frac{\pi}{6}s}\sin\tfrac{\pi}{6}$$

$$(2s^2+s+4)Y = e^{-\frac{\pi}{6}s}$$

$$Y = \frac{e^{-\frac{\pi}{6}s}}{2s^2+s+4}$$

$$y = \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{6}s}}{2s^2+s+4}\right\}$$

$$= \frac{1}{2}\,\mathcal{U}_{\frac{\pi}{6}}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{1}{2}s+2}\right\}_{t\to t-\frac{\pi}{6}}$$

$$= \frac{1}{2}\,\mathcal{U}_{\frac{\pi}{6}}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2+\frac{31}{16}}\right\}_{t\to t-\frac{\pi}{6}}$$

$$= \frac{1}{2}\,\mathcal{U}_{\frac{\pi}{6}}(t) - \frac{4}{\sqrt{31}}\,\mathcal{L}^{-1}\left\{ \frac{\frac{\sqrt{31}}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{31}}{4}\right)^2} \right\}_{t\to t-\frac{\pi}{6}}$$

$$= \frac{2}{\sqrt{31}}\,\mathcal{U}_{\frac{\pi}{6}}(t) \left(e^{-\frac{1}{4}t} \sin\left(\frac{\sqrt{31}}{4}t\right) \right)_{t\to t-\frac{\pi}{6}}$$

$$= \frac{2}{\sqrt{31}}\,\mathcal{U}_{\frac{\pi}{6}}(t)\, e^{-\frac{1}{4}\left(t-\frac{\pi}{6}\right)} \sin\left(\frac{\sqrt{31}}{4}\left(t-\frac{\pi}{6}\right)\right).$$

Ex. (H.W's) Solve the DEs.

① $\begin{cases} y'' + 3y' + 2y = \delta(t-5) + \mathcal{U}_{10}(t) \\ y(0) = y'(0) = 0 \end{cases}$

② $\begin{cases} y'' + y = \delta(t-2\pi)\cos t \\ y(0) = y'(0) = 1. \end{cases}$

③ $\begin{cases} y'' + y' + y = \delta(t-\pi)\cos t + \mathcal{U}_1(t) \\ y(0) = y'(0) = 0. \end{cases}$

④ $\begin{cases} y'' = t^2\,\delta(t-2) \\ y(0) = 0,\ y'(0) = 1. \end{cases}$

6.6 The Convolution Integrals

Df: If the functions f and g are piecewise continuous on $[0,\infty)$, then $f*g$ is defined by $f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$ and is called the convolution of f and g.

The convolution $f*g$ is a function of t.

Ex. Find $t * \sin t$.

Sol. $t*\sin t = \int_0^t \tau \sin(t-\tau)\, d\tau$.

$$= \tau\cos(t-\tau) + \sin(t-\tau) \Big]_{\tau=0}^{\tau=t}$$

τ		$\sin(t-\tau)$
	$(+)$	$\cos(t-\tau)$
1	$(-)$	
0		$-\sin(t-\tau)$

$$= \left[t\cos(t-t) + \sin(t-t)\right] - \left[0\cdot\cos(t-0) + \sin(t-0)\right]$$

$$= t - \sin t.$$

Ex. $t * e^t = \int_0^t (t-\tau)\, e^\tau \, d\tau$

$$u = t - \tau \qquad dv = e^\tau \, d\tau$$

$$du = -d\tau \qquad v = e^\tau$$

$$\therefore \; t * e^t = (t-\tau)\, e^\tau \Big|_{\tau=0}^{\tau=t} + \int_0^t e^\tau \, d\tau$$

$$= (t-t)\, e^t - t e^0 + e^\tau \Big|_{\tau=0}^{\tau=t}$$

$$= -t + e^t - e^0 = -t + e^t - 1.$$

Thm (Convolution Theorem)

If f and g are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\}$$
$$= F(s)\, G(s)$$

OR $\mathcal{L}^{-1}\{F(s)\, G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$

$$= (f * g)(t).$$

Ex. $\mathcal{L}\{e^t * \sin t\} = \mathcal{L}\{e^t\}\,\mathcal{L}\{\sin t\}$

$= \frac{1}{s-1} \cdot \frac{1}{s^2+1}$.

ex. $\mathcal{L}\left\{\int_0^t \tau \sin(t-\tau)\, d\tau\right\}$

$= \mathcal{L}\{t * \sin t\}$

$= \mathcal{L}\{t\}\,\mathcal{L}\{\sin t\}$

$= \frac{1}{s^2} \cdot \frac{1}{s^2+1}$.

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+16)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+16} \cdot \frac{1}{s^2+16}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$

$= \frac{1}{4}\sin 4t * \frac{1}{4}\sin 4t$

$= \frac{1}{16}\int_0^t \sin(4\tau)\,\sin(4(t-\tau))\, d\tau$

(214)

We use the identity

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right].$$

$$= \frac{1}{16}\cdot\frac{1}{2}\int_0^t \left[\cos(4\tau - 4t + 4\tau) - \cos(4\tau + 4t - 4\tau)\right].$$

$$= \frac{1}{32}\int_0^t \left[\cos(8\tau - 4t) - \cos(4t)\right] d\tau$$

$$= \frac{1}{32}\left[\frac{\sin(8\tau - 4t)}{8} - \tau\cos 4t\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{32}\left[\frac{\sin(8t-4t)}{8} - t\cos 4t - \frac{\sin(-4t)}{8} + 0\right]$$

$$= \frac{1}{32}\left[\frac{\sin 4t}{8} - t\cos 4t + \frac{\sin 4t}{8}\right]$$

$$= \frac{1}{32}\left[\frac{\sin 4t}{4} - t\cos 4t\right]$$

$$= \frac{\sin 4t - 4t\cos 4t}{128}.$$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$$= 1 * \sin t$$

$$= \int_0^t 1.\ \sin(t-\tau)\, d\tau$$

$$= \frac{-\cos(t-\tau)}{-}\Bigg|_{\tau=0}^{\tau=t}$$

$$= \cos(t-t) - \cos(t-0)$$

$$= 1 - \cos t .$$

<u>Application.</u>

Ex. Solve the integral Eq.

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)\, e^{t-\tau}\, d\tau$$

<u>Sol.</u> Take $\mathcal{L}.T$:

$$\mathcal{L}\{f(t)\} = 3\,\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\} - \mathcal{L}\{f * e^{t}\}$$

$$F(s) = 3 \cdot \frac{2!}{s^3} - \frac{1}{s+1} - F \cdot \frac{1}{s-1}$$

$$\left(1 + \frac{1}{s-1}\right) F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

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$$\frac{s}{s-1}F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$F(s) = \frac{s-1}{s}\left[\frac{6}{s^3} - \frac{1}{s+1}\right]$$

$$= \frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)}$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} - \frac{s-1}{s(s+1)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{s-1}{s(s+1)}\right\}$$

Now, $\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

$$s-1 = A(s+1) + Bs$$

$s=0 \Rightarrow \boxed{-1 = A}$, $s=-1 \Rightarrow -2 = -B$, $\boxed{B=2}$

$$\therefore f(t) = 3\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{-1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 3t^2 - t^3 + 1 - 2e^{-t}.$$

Ex. Solve the following integro-differential Eq.

$$y'(t) - \frac{1}{2}\int_0^t (t-\tau)^2 y(\tau)\,d\tau = -t, \quad y(0) = 1.$$

Sol. Take L.T:

$$\mathcal{L}\{y'\} - \frac{1}{2}\mathcal{L}\{t^2 * y\} = -\mathcal{L}\{t\}$$

$$sY - 1 - \frac{1}{2}\mathcal{L}\{t^2\}\,\mathcal{L}\{y\} = \frac{-1}{s^2}$$

$$sY - 1 - \frac{1}{2}\cdot\frac{2!}{s^3}Y = \frac{-1}{s^2}$$

$$\left(s - \frac{1}{s^3}\right)Y = 1 - \frac{1}{s^2}$$

$$\frac{s^4 - 1}{s^3}Y = \frac{s^2 - 1}{s^2}$$

$$Y = \frac{s^2-1}{s^2}\cdot\frac{s^3}{s^4-1} = \frac{(s^2-1)\,s^3}{s^2(s^2-1)(s^2+1)}$$

$$\Rightarrow Y = \frac{s}{s^2+1}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

Ex. Solve the integro-differential eq.

$$\begin{cases} \phi'(t) + \phi(t) = \int_0^t \sin(t-\tau)\,\phi(\tau)\,d\tau, \\ \phi(0) = 1 \end{cases}$$

Sol. Take L.T :

$$\mathcal{L}\{\phi'(t)\} + \mathcal{L}\{\phi(t)\} = \mathcal{L}\{\sin t * \phi(t)\}$$

$$s\,\Phi - \phi(0) + \Phi = \frac{1}{s^2+1}\,\Phi$$

$$\left(s + 1 - \frac{1}{s^2+1}\right)\Phi = 1$$

$$\frac{(s+1)(s^2+1) - 1}{s^2+1}\,\Phi = 1$$

$$\Rightarrow \quad \Phi = \frac{s^2+1}{s^3+s+s^2} = \frac{s^2+1}{s(s^2+s+1)}$$

$$\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s^2+s+1)}\right\}$$

Now,

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$$\frac{s^2+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$s^2+1 = A(s^2+s+1) + (Bs+C)s$$

$$s^2+1 = As^2+As+A + Bs^2+Cs$$

$$s^2+1 = (A+B)s^2 + (A+C)s + A$$

$$\Rightarrow A+B=1,\ A+C=0,\ A=1$$

$$\therefore B=0,\ C=-1$$

$$\therefore \phi(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\}$$

$$= 1 - \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$= 1 - \sqrt{\frac{4}{3}}\ \mathcal{L}^{-1}\left\{\frac{\sqrt{\frac{3}{4}}}{(s+\frac{1}{2})^2+\left(\sqrt{\frac{3}{4}}\right)^2}\right\}$$

$$\phi(t) = 1 - \frac{2}{\sqrt{3}}\, e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

Ex. (H.W's) Solve

① $y(t) + 2\int_0^t \cos(t-\tau)\, y(\tau)\, d\tau = e^{-t}$.

② $y(t) + \int_0^t (t-\tau)\, y(\tau)\, d\tau = \sin 2t$.

③ $y' + 2y = \int_0^t y(\tau)\, d\tau$, $y(0) = 1$.

④ $y + 2\int_0^t y(\tau) \cos(t-\tau)\, d\tau = 4e^{-t} + \sin t$.

7.5 Homogeneous linear systems with constant coefficients.

In this section, we will study systems of homogeneous linear equations with constant coefficients, that is system of the form

$$\underbrace{\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} \quad (*)$$

or $\quad X' = AX,$

where $\quad x_i'(t) = \dfrac{dx_i}{dt}, \quad i = 1, 2, \dots, n.$

We focus on 2×2-system, that is,

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a, b, c, d are constants.

We will assume that $X(t) = Ke^{rt}$ is

a solution of (*). then, $X' = rKe^{rt}$

Hence, $rKe^{rt} = AKe^{rt}$

$\Rightarrow \boxed{(A - rI)K = 0}$ (**), where

$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, I is the $n\times n$-identity matrix

the system (**) has nontrivial solution $K \neq 0$ when the matrix $A - rI$ is singular

that is $\boxed{\det(A - rI) = 0}$ (***) To solve the system (*), we must solve the system of algebraic eq (**).

r in eq (**) is called the eigenvalue of A and K is called the corresponding eigenvector. In this section, we study the case when r is real eigenvalues and distinct.

The eq. (***) is called the characteristic eq.

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ex1 Solve the system $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$.

Sol. step ① the characterstic eq. is

$\det(A - rI) = 0$

$\Rightarrow \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \Rightarrow (1-r)^2 - 4 = 0$

$\Rightarrow r^2 - 2r - 3 = 0$

$\Rightarrow (r-3)(r+1) = 0 \Rightarrow \boxed{r_1 = 3}, \boxed{r_2 = -1}$

are the eigenvalues of A.

step ② let $K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector

corresponding to $\boxed{r_1 = 3}$. Then

$(A - r_1 I) K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (A - 3I) K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -2k_1 + k_2 = 0 \Rightarrow k_2 = 2k_1$

$4k_1 - 2k_2 = 0 \Rightarrow k_2 = 2k_1$

Take $\boxed{k_1 = 1} \Rightarrow \boxed{k_2 = 2}$

$\Rightarrow$ $\boxed{K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ is an eigenvector corresponding to $\boxed{r_1 = 3}$

For $\boxed{r_2 = -1}$, let $K_2 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector corresponding to $\boxed{r_2 = -1}$

then $(A - r_2 I) K_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{ll} 2k_1 + k_2 = 0 \Rightarrow k_2 = -2k_1 \\ 4k_1 + 2k_2 = 0 \Rightarrow k_2 = -2k_1 \end{array}$$

Take $k_1 = 1 \Rightarrow k_2 = -2$.

$\Rightarrow$ $\boxed{K_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ is an eigenvector corresponding to $\boxed{r_2 = -1}$

We conclude that the general solution of the system is

$$X = c_1 X_1 + c_2 X_2 = c_1 K_1 e^{r_1 t} + c_2 K_2 e^{r_2 t}$$

$$\Rightarrow X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Ex 2 Solve

$$\frac{dx}{dt} = 5x - y \,, \qquad x(0) = 2$$

$$\frac{dy}{dt} = 3x + y \qquad y(0) = -1$$

Sol. we first write the system in a matrix form

$$X' = AX \,, \quad X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$.

the characterstic eq. is

$$\det(A - rI) = 0 \Rightarrow \begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = 0$$

$$\Rightarrow (5-r)(1-r) + 3 = 0$$

$$\Rightarrow r^2 - 6r + 8 = 0$$

$$\Rightarrow (r-2)(r-4) = 0$$

$\Rightarrow \boxed{r_1 = 2}$, $\boxed{r_2 = 4}$ are the eigenvalues

Let $K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigen vector corresponding to $\boxed{r_1 = 2}$. Then

$$(A - 2I)K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3k_1 - k_2 = 0$$

$$\Rightarrow \boxed{k_2 = 3k_1}$$

Take $\boxed{k_1 = 1} \Rightarrow \boxed{k_2 = 3}$

$\therefore$ $\boxed{K_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$ is an eigen vector corr. to $\boxed{r_1 = 2}$

For $\boxed{r_2 = 4}$, let $K_2 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigen vector corresp. to $r_1 = 4$. Then,

$$(A - 4I)K_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow k_1 = k_2$. Take $k_2 = 1 \Rightarrow k_1 = 1$

$\therefore$ $\boxed{K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is an eigen vector corr. to $\boxed{r_2 = 4}$

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$\therefore$ the general solution of the system is

$$X = c_1 e^{r_1 t} K_1 + c_2 e^{r_2 t} K_2$$

$$\boxed{X = c_1 e^{2t}\begin{pmatrix}1\\3\end{pmatrix} + c_2 e^{4t}\begin{pmatrix}1\\1\end{pmatrix}}$$

now, $X(0) = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

$$\Rightarrow c_1 + c_2 = 2$$

$$-1\,(3c_1 + c_2 = -1$$

$$-2c_1 = 3 \Rightarrow c_1 = -3/2$$

$$c_2 = 7/2$$

$$\therefore X = -\tfrac{3}{2} e^{2t}\begin{pmatrix}1\\3\end{pmatrix} + \tfrac{7}{2} e^{4t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-\tfrac{3}{2} e^{2t} + \tfrac{7}{2} e^{4t}\\ -\tfrac{9}{2} e^{2t} + \tfrac{7}{2} e^{4t}\end{pmatrix}$$

$\blacksquare$

7.6 Complex Eigenvalues

Ex1. Solve the IVP

$$X' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Sol. the characterstic equation is

$\det(A - rI) = 0$, where $A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} 2-r & 8 \\ -1 & -2-r \end{vmatrix} = 0$$

$$\Rightarrow (2-r)(-2-r) + 8 = 0$$

$$\Rightarrow -4 - 2r + 2r + r^2 + 8 = 0$$

$$\Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$\Rightarrow \boxed{r_1 = 2i}$, $\boxed{r_2 = \bar{r_1} = -2i}$ are the eigenvalues.

For $r_1 = 2i$, let $K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector corresponding to $r_1 = 2i$. Then

$$(A - r_1 I)K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 2-2i & 8 \\ -1 & -2-2i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (2-2i)k_1 + 8k_2 = 0$$

$$-k_1 - (2+2i)k_2 = 0$$

$$\Rightarrow \boxed{k_1 = -(2+2i)k_2}$$

By choosing $k_2 = -1$, we get $k_1 = 2+2i$

$\Rightarrow K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix}$ is an eigenvector

Corresponding to $r_1 = 2i$

$$\Rightarrow X = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} e^{(2i)t}$$

$$= \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} (2+2i)(\cos 2t + i \sin 2t) \\ -\cos 2t - i \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos 2t - 2\sin 2t + i(2\cos 2t + 2\sin 2t) \\ -\cos 2t - i \sin 2t \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix}}_{X_1} + i \underbrace{\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}}_{X_2}$$

$$\therefore X = c_1 X_1 + c_2 X_2$$

$$\therefore X = c_1 \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}.$$

$$X(0) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 2c_1 + 2c_2 &= 2 \\ -c_1 &= -1 \end{aligned} \Rightarrow \boxed{c_1 = 1}, \boxed{c_2 = 0}$$

$$\therefore X(t) = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix}.$$

<u>Ex 2.</u> Solve $\begin{cases} \dfrac{dx}{dt} = -\frac{1}{2}x + y \\ \dfrac{dy}{dt} = -x - \frac{1}{2}y \end{cases}$

<u>Sol.</u> $X' = \underbrace{\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}}_{A} X$, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

The characteristic eq. is $|A - rI| = 0$

$$\Rightarrow \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = 0 \Rightarrow \left(-\frac{1}{2} - r\right)^2 + 1 = 0$$

$$\Rightarrow \frac{1}{2} + r = \pm i \Rightarrow \boxed{r = -\frac{1}{2} \pm i}$$

$\Rightarrow r_1 = -\frac{1}{2}+i, \quad r_2 = \overline{r_1} = -\frac{1}{2}-i$

For $\boxed{r_1 = -\frac{1}{2}+i}$, Let $K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector corresponding to $r_1 = -\frac{1}{2}+i$, then solve the system $(A - r_1 I) K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -\frac{1}{2}-r_1 & 1 \\ -1 & -\frac{1}{2}-r_1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -ik_1 + k_2 = 0, \quad -k_1 - ik_2 = 0$$

$$\Rightarrow \boxed{k_2 = ik_1}$$

By choosing $k_1 = 1$, we get $k_2 = i$

$\Rightarrow K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector corresponding to $r_1 = -\frac{1}{2}+i$.

$$X = K_1 e^{r_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}t} \cdot e^{it}$$

$$= \begin{pmatrix} e^{-\frac{1}{2}t}(\cos t + i\sin t) \\ i e^{-\frac{1}{2}t}(\cos t + i\sin t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{1}{2}t}\cos t + i e^{-\frac{1}{2}t}\sin t \\ -e^{-\frac{1}{2}t}\sin t + i e^{-\frac{1}{2}t}\cos t \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{-\frac{1}{2}t}\cos t \\ -e^{-\frac{1}{2}t}\sin t \end{pmatrix}}_{X_1} + i \underbrace{\begin{pmatrix} e^{-\frac{1}{2}t}\sin t \\ e^{-\frac{1}{2}t}\cos t \end{pmatrix}}_{X_2}$$

$$X = c_1 X_1 + c_2 X_2$$

$$= c_1 \begin{pmatrix} e^{-\frac{1}{2}t}\cos t \\ -e^{-\frac{1}{2}t}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-\frac{1}{2}t}\sin t \\ e^{-\frac{1}{2}t}\cos t \end{pmatrix}$$

$$= c_1 e^{-\frac{1}{2}t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Note: $W(X_1, X_2) = \begin{vmatrix} e^{-\frac{1}{2}t}\cos t & e^{-\frac{1}{2}t}\sin t \\ -e^{-\frac{1}{2}t}\sin t & e^{-\frac{1}{2}t}\cos t \end{vmatrix}$

$$= e^{-t}(\cos^2 t + \sin^2 t)$$

$$= e^{-t} \neq 0 \Rightarrow \{X_1, X_2\}$$ form a fundamental set of solutions.

7.8 Repeated Eigenvalues

Ex 1 Find a fundamental set of solutions of

$$X' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X.$$

Sol. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

The characterstic eq. is $|A - rI| = 0$

$$\Rightarrow \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = 0 \Rightarrow (1-r)(3-r)+1=0$$

$$\Rightarrow 3 - 4r + r^2 + 1 = 0$$

$$\Rightarrow r^2 - 4r + 4 = 0$$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r_1 = r_2 = 2.$$

$\therefore r = 2$ is a double eigenvalue.

Let $K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector corresponding to $r = 2$. Then solve $(A - 2I)K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{k_1 = -k_2}$$

choose $k_2 = 1 \Rightarrow k_1 = -1$

$\Rightarrow K = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $r_1 = r_2 = 2$ $\Rightarrow$ $\boxed{X_1 = K e^{2t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}}$

But there is no second solution of the form $X = K e^{rt}$.

To find a second solution, let $\boxed{X = K t e^{2t}}$ where K is a constant vector to be determined.

$$X' = K e^{2t} + 2 K t e^{2t}$$

Substitute for X in $X' = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}_{A} X$:

$$\boxed{K e^{2t} + 2 K t e^{2t} - A K t e^{2t} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \quad (*)$$

$\Rightarrow$ For Eq $(*)$ to be satisfied for all t, it is necessary for the coefficients of $t e^{2t}$ and e^{2t} both to be zero.

e^{2t} terms: $K = 0$

~~te^{2t} terms:~~

Hence there is no nonzero solution of the system $X' = AX$ of the form $X = Kte^{2t}$

We assume $\boxed{X = Kte^{2t} + Pe^{2t}}$, (**)

where K and P are constant vector to be determined.

Substitute (**) in $X' = AX$.

$$X' = Ke^{2t} + 2Kte^{2t} + 2Pe^{2t}$$

$$\Rightarrow 2Kte^{2t} + (K + 2P)e^{2t} = A(Kte^{2t} + Pe^{2t})$$

te^{2t}: $\quad 2K = AK \Rightarrow (A - 2I)K = 0$

e^{2t}: $\quad K + 2P = AP \Rightarrow (A - 2I)P = K$

Hence, to find the second solution, solve

$$(A - 2I)P = K$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boxed{-P_1 - P_2 = 1}$$

So, if $P_2 = \eta$, where η is arbitrary, then

$$\boxed{P_1 = -\eta - 1}.$$

$$\Rightarrow P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \eta \\ -\eta-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Now, Substitute for K and P into $(**)$

$$X = K t e^{2t} + P e^{2t}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \eta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

multiple of X_1 (absorb).

$$\therefore \boxed{X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}}$$ is the second solution. and the general sol. is

$$X = c_1 X_1 + c_2 X_2$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$$

Ex2. Solve $X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X$.

Sol. $A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$

the characterstic eq. is $|A - rI| = 0$.

$$\Rightarrow \begin{vmatrix} 3-r & -18 \\ 2 & -9-r \end{vmatrix} = 0$$

$$\Rightarrow (3-r)(-9-r) + 36 = 0$$

$$\Rightarrow -27 - 3r + 9r + r^2 + 36 = 0$$

$$\Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0$$

$$\Rightarrow \boxed{r_1 = r_2 = -3}$$

$\therefore r = -3$ is a double eigen value.

Let $K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be an eigenvector corresponding to $r = -3$. then solve $(A + 3I)K = 0$

i.e., $\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} 6k_1 - 18k_2 = 0 \\ 2k_1 - 6k_2 = 0 \end{cases} \Rightarrow \boxed{k_1 = 3k_2}$$

choose $k_2 = 1$, we get $k_1 = 3$

$$\Rightarrow K = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \boxed{X_1 = Ke^{-3t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}}$$

$\boxed{X_2 = K t e^{-3t} + P e^{-3t}}$, where P is a solution of $(A+3I)P = K$

$$\Rightarrow \begin{pmatrix} 3+3 & -18 \\ 2 & -9+3 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 6P_1 - 18P_2 = 3 \\ 2P_1 - 6P_2 = 1 \end{matrix} \Rightarrow \boxed{P_1 = \tfrac{1}{2} + 3P_2}$$

Let $P_2 = \eta$ (arbitrary)

$$\Rightarrow P_1 = \tfrac{1}{2} + 3\eta$$

$$\therefore P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 3\eta \\ \eta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \eta \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\therefore X = K t e^{-3t} + P e^{-3t}$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \left[\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \eta \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right] e^{-3t}$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} + \eta \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}}_{\text{multiple of } X_1 \text{ (absorb).}}$$

$$\therefore \boxed{X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}}$$

$$\therefore X = c_1 X_1 + c_2 X_2$$

$$= c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right]$$