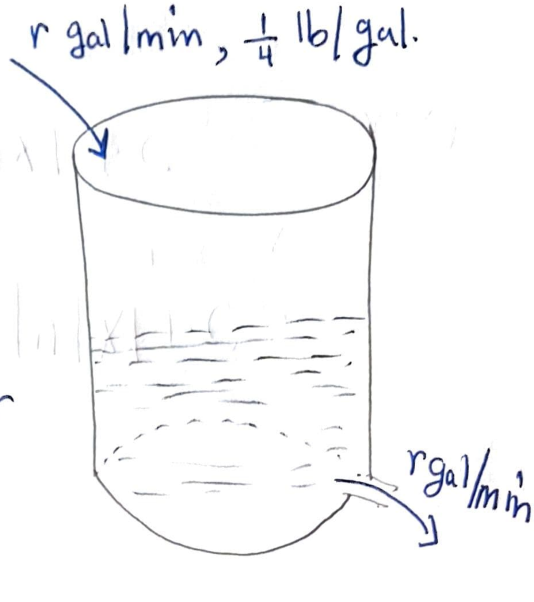
2.3 Modeling with first Order Equations



Ex. 1 Mixing problem

At $t=0$, Q_0 lb of Salt dissolved in 100 gal of water

$Q(t)$: quantity of salt in the tank at time t.

Setup an IVP that describes the flow process
≡ Diff. Eq. for $Q(t)$. Then find $Q(t)$.

$$\frac{dQ}{dt} = (\text{rate in of salt}) - (\text{rate out of salt}) \quad \text{lb/min}$$

$$\frac{dQ}{dt} = \frac{1}{4}\frac{\text{lb}}{\cancel{\text{gal}}} * r\frac{\cancel{\text{gal}}}{\text{min}} - \frac{Q(t)}{100}\frac{\text{lb}}{\cancel{\text{gal}}} * r\frac{\cancel{\text{gal}}}{\text{min}}$$

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{r}{100}Q(t), \quad Q(0) = Q_0$$

$$\Rightarrow \frac{dQ}{dt} + \frac{r}{100}Q(t) = \frac{r}{4}, \quad Q(0) = Q_0.$$

Limiting quantity (equilibrium) Q :

Setting $\frac{dQ}{dt} = 0 \Rightarrow \frac{r}{100} Q_L = \frac{r}{4} \Rightarrow Q_L = 25$ lb.

Solution of the IVP

$$\frac{dQ}{dt} + \frac{r}{100} Q(t) = \frac{r}{4}, \quad Q(0) = Q_0$$

$$\mu(t) = e^{\frac{r}{100}t}$$

$$Q(t) = \frac{1}{e^{\frac{r}{100}t}} \left[\int \frac{r}{4} e^{\frac{r}{100}t} \, dt + c \right]$$

$$= \frac{1}{e^{\frac{r}{100}t}} \left[\frac{r}{4} \frac{100}{r} e^{\frac{r}{100}t} + c \right]$$

$$Q(t) = 25 + c e^{-\frac{r}{100}t}$$

$$Q(0) = Q_0 = 25 + C \Rightarrow c = Q_0 - 25$$

$$Q(t) = 25 + (Q_0 - 25) e^{-\frac{r}{100}t}$$

$$\lim_{t \to \infty} Q(t) = 25 = Q_L.$$

let $Q_0 = 2Q_L = 50$, $r = 3$.

$$Q(t) = 25 + 25e^{-0.03t}$$

Find $t = T$ such that $Q = 25.5$

$$25.5 = 25 + 25e^{-0.03T}$$

$$0.5 = 25e^{-0.03T}$$

$$\frac{1}{50} = e^{-0.03T} \Rightarrow \ln 50 = 0.03T$$

$$T = \frac{\ln 50}{0.03} \cong 130.4 \text{ min}.$$

Find r such that $T = 45$ min, $Q = 25.5$

$$25.5 = 25 + 25e^{-\frac{r}{100}(45)}$$

$$\frac{1}{50} = e^{-0.45r} \Rightarrow \ln \frac{1}{50} = -0.45r$$

$$\ln 50 = 0.45r$$

$$r = \frac{\ln 50}{0.45} \cong 8.69 \text{ gal/min}.$$

Ex.2 Continuous interest rate

A sum of money S_0 is deposited in a bank at $t=0$, that pays interest at a rate r annually. Suppose that deposits or withdrawals take place at a constant rate k. Write an IVP for $S(t)$.

$$\frac{dS}{dt} = rS(t) + k, \quad S(0) = S_0, \quad \begin{array}{l} k>0 \text{ deposit} \\ k<0 \text{ withdrawals} \end{array}$$

$$\frac{dS}{dt} - rS = k, \quad S(0) = S_0$$

$$\mu(t) = e^{-rt}$$

$$S(t) = \frac{1}{e^{-rt}}\left[\int k e^{-rt}\,dt + c\right]$$

$$= \frac{1}{e^{-rt}}\left[-\frac{k}{r}e^{-rt} + c\right]$$

$$S(t) = -\frac{k}{r} + ce^{+rt}$$

$$S(0) = S_0 = -\frac{k}{r} + c \Rightarrow c = S_0 + \frac{k}{r}$$

$$S(t) = -\frac{k}{r} + \left(S_0 + \frac{k}{r}\right)e^{rt}.$$

Solution of Sec. 2.3

4. A tank with Capacity 500 L originally Contains 200 L of water with 100 kg of salt.

Setup and solve an IVP for $Q(t)$.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dQ}{dt} = (3)(1) - \frac{Q(t)}{200+t} * 2$$

$$\frac{dQ}{dt} = 3 - \frac{2}{200+t} Q(t) \quad , \quad Q(0) = 100.$$

$$\frac{dQ}{dt} + \frac{2}{200+t} Q(t) = 3 \quad , \quad Q(0) = 100$$

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = e^{2\ln(200+t)} = (200+t)^2$$

$$Q(t) = \frac{1}{(200+t)^2} \left[\int 3(200+t)^2 dt + C \right]$$

$$= \frac{1}{(200+t)^2} \left[(200+t)^3 + C \right]$$

$$Q(t) = 200 + t + \frac{c}{(200+t)^2},$$

$$Q(0) = 100 = 200 + \frac{c}{(200)^2} \Rightarrow c = -4 \times 10^6$$

$$Q(t) = 200+t - \frac{4 \times 10^6}{(200+t)^2}$$

Find $Q(t)$ just before the tank overflows

$$Q(300) = 500 - \frac{4 \times 10^6}{(500)^2}$$

$$\text{Concentration} = \frac{Q(t)}{200+t} = 1 - \frac{4 \times 10^6}{(200+t)^3}$$

If the tank has infinite capacity

$$\lim_{t \to \infty} \frac{Q(t)}{200+t} = \lim_{t \to \infty} 1 - \frac{4 \times 10^6}{(200+t)^3}$$

$$= 1.$$

18. Newton's Law of Cooling / heating

$$\frac{du}{dt} = -k\left[u - T(t)\right]$$

where $T(t) = T_0 + T_1 \cos \omega t$ is the ambient temper.

$$\frac{du}{dt} = -k\left[u - T_0 - T_1 \cos \omega t\right]$$

$$\frac{du}{dt} + ku(t) = kT_0 + kT_1 \cos(\omega t)$$

$$\mu(t) = e^{kt}$$

$$u(t) = \frac{1}{e^{kt}}\left[\int kT_0 e^{kt}\,dt + kT_1 \int e^{kt} \cos \omega t\,dt + c\right]$$

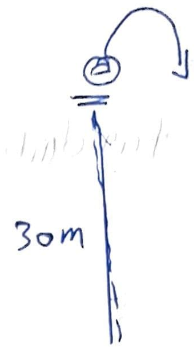
$$= \frac{1}{e^{kt}}\left[T_0 e^{kt} + kT_1 \int e^{kt} \cos \omega t\,dt + c\right]$$

Solving $\int e^{kt} \cos(\omega t)\,dt$ by parts, we get

$$\int e^{kt} \cos \omega t\,dt = \frac{\omega}{k^2+\omega^2} e^{kt}\left[\sin(\omega t) + \frac{k}{\omega} \cos(\omega t)\right]$$

$$u(t) = T_0 + \frac{T_1 \omega}{k^2+\omega^2}\left(\sin(\omega t) + \frac{k}{\omega}\cos(\omega t)\right) + c e^{-kt}$$

20. $m = 0.15\,kg$, $v(0) = 20\,m/s$



$$ma = F$$

$$m\frac{dv}{dt} = -mg$$

$$\frac{dv}{dt} = -g$$

$$v(t) = -gt + c \Rightarrow v(t) = -9.8t + c$$

$$v(0) = 20 = c$$

$$v(t) = -9.8t + 20$$

$$\frac{ds}{dt} = -9.8t + 20$$

$$s(t) = -4.9t^2 + 20t + c_1$$

$$s(0) = 30 = c$$

$$s(t) = -4.9t^2 + 20t + 30$$

max. height when $v(t) = 0 \Rightarrow t = \frac{20}{9.8} \approx 2.041\,s$

$$s(2.041) \simeq 50.41$$

$$s(t) = 0 = -4.9t^2 + 20t + 30 \Rightarrow t \approx 5.428\,s$$