Ch. 4 Higher Order Diff. Equations

4.2 + 4.3 Homog. Eqs + Undetermined Coeffs.

Consider the n^{th} order diff. eq. with Constant Coeff

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n = 0$$

$y = e^{rt}$ is a solution $\Longleftrightarrow$

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

This is an n^{th} order polynomial which has n roots:

1. If $r_1, \ldots, r_n$ are real, distinct roots the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}.$$

Ex. Find the general solution of the I.V.P.

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = -1.$$

The characteristic polynomial is:

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

by substitution $r = \pm 1$ are solutions

$$r^4 + r^3 - 7r^2 - r + 6$$

$$= (r^2 - 1)(r^2 + r - 6)$$

$$= (r-1)(r+1)(r+3)(r-2)$$

$$\begin{array}{r|l} & r^2 + r - 6 \\ \hline r^2 - 1 & r^4 + r^3 - 7r^2 - r + 6 \\ & r^4 - r^2 \\ \hline & r^3 - 6r^2 - r + 6 \\ & r^3 - r \\ \hline & -6r^2 + 6 \\ & -6r^2 + 6 \\ \hline & -- \end{array}$$

$r_1 = 1, \; r_2 = -1, \; r_3 = -3, \; r_4 = 2$

general solution

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-3t} + c_4 e^{2t}$$

ICs:

$$c_1 + c_2 + c_3 + c_4 = 1$$

$$c_1 - c_2 - 3c_3 + 2c_4 = 0$$

$$c_1 + c_2 + 9c_3 + 4c_4 = -2$$

$$c_1 - c_2 - 27c_3 + 8c_4 = -1$$

Rem. If $r=\frac{p}{q}$ is a rational root of the polynomial

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

then, p must be a factor of a_n and
q must be a factor of a_0

Ex. $r^4 + r^3 - 7r^2 - r + 6 = 0$

$r=\frac{p}{q}$, p must be a factor of 6 : $\pm 1, \pm 2, \pm 3, \pm 6$
q must be a factor of 1 : ± 1,

* Complex roots:

If $r = \lambda \mp i\mu$ is repeated s times

$$\left.\begin{matrix} \lambda + i\mu \\ \lambda - i\mu \end{matrix} \begin{matrix} \text{s-times} \\ \text{s-times} \end{matrix}\right\} \text{2s-times}$$

gives the 2s-solutions

$$\left.\begin{matrix} e^{\lambda t}\sin \mu t, & e^{\lambda t}\cos(\mu t) \\ t e^{\lambda t}\sin(\mu t), & t e^{\lambda t}\cos(\mu t) \\ \vdots & \\ t^{s-1} e^{\lambda t}\sin(\mu t), & t^{s-1} e^{\lambda t}\cos(\mu t) \end{matrix}\right\} \text{2s-functions}$$

* Repeated real root:

If r is a repeated real root s-times:

$e^{rt}, \; te^{rt}, \; \ldots, \; t^{s-1}e^{rt}.$

Ex. $y^{(4)} - y = 0.$

$r^4 - 1 = 0 \Leftrightarrow (r^2-1)(r^2+1) = 0$

$(r-1)(r+1)(r^2+1) = 0$

$r_1 = 1, \; r_2 = -1, \; r_3 = i, \; r_4 = -i$

$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \sin t + c_4 \cos t.$

Ex. Find Y_p for $y^{(4)} - y = e^t + \sin t$

$Y_p(t) = Ate^t + Bt\sin t + Ct\cos t.$

Ex. Find the general solution of $y''' + y = 0$

$r^3 + 1 = 0 \Leftrightarrow (r+1)(r^2 - r + 1) = 0$

$r = -1, \; r = \frac{1 \mp \sqrt{3}i}{2} = \frac{1}{2} \mp i\frac{\sqrt{3}}{2}$

$y(t) = c_1 e^{-t} + c_2 e^{t/2} \sin \frac{\sqrt{3}}{2}t + c_3 e^{t/2} \cos \frac{\sqrt{3}}{2}t.$

Ex. Solve $y^{(4)} + 2y'' + y = 0$.

$$r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2+1)^2 = 0$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i, \pm i$$

$$y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t$$

Ex. Find the general form of $Y_p(t)$ for

$$y^{(4)} + 2y'' + y = \cos t$$

$$Y_p(t) = At^2 \cos t + Bt^2 \sin t.$$

Ex. Find the general solution of $y''' - 3y'' + 3y' - y = 4e^t$.

$$y''' - 3y'' + 3y' - y = 0$$

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$$

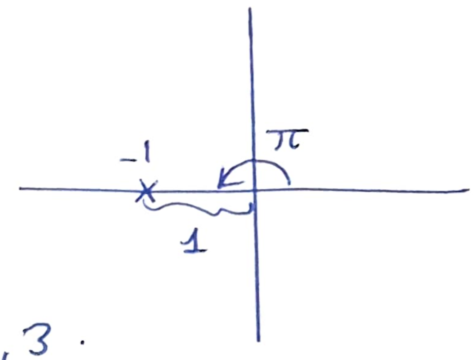
$$y_c(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t.$$

$$Y_p(t) = At^3 e^t$$

$$y(t) = Y_p(t) + y_c(t).$$

Ex. Find the general solution of $y^{(4)} + y = 0$.

$$r^4 + 1 = 0 \Rightarrow r^4 = -1$$



$$r^4 = -1 = 1\, e^{i(\pi + 2k\pi)}$$

$$r_k = e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)}, \quad k = 0, 1, 2, 3.$$

$$r_0 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

$$r_1 = e^{i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$r_2 = e^{i\left(\frac{\pi}{4} + \pi\right)} = e^{i\pi} e^{i\frac{\pi}{4}} = -e^{i\frac{\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$

$$r_3 = e^{i\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)} = e^{i\frac{7\pi}{4}} = e^{i\left(2\pi - \frac{\pi}{4}\right)} = e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} \mp i\frac{1}{\sqrt{2}} \longrightarrow e^{t/\sqrt{2}}\cos\frac{t}{\sqrt{2}},\ e^{t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}$$

$$-\frac{1}{\sqrt{2}} \mp i\frac{1}{\sqrt{2}} \longrightarrow e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}},\ e^{-\frac{t}{\sqrt{2}}}\sin\frac{t}{\sqrt{2}}$$

general solution,

$$y(t) = e^{\frac{t}{\sqrt{2}}}\left(c_1\cos\frac{t}{\sqrt{2}} + c_2\sin\frac{t}{\sqrt{2}}\right)$$
$$+ e^{-\frac{t}{\sqrt{2}}}\left(c_3\cos\frac{t}{\sqrt{2}} + c_4\sin\frac{t}{\sqrt{2}}\right).$$

4.2

(14) $y^{(4)} - 4y''' + 4y'' = 0$

$r^4 - 4r^3 + 4r^2 = 0 \iff r^2(r^2 - 4r + 4) = 0$

$r^2(r-2)^2 = 0 \Rightarrow r_1 = 0,\ r_2 = 0,\ r_3 = r_4 = 2$

$y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$.

18. $y^{(6)} - y'' = 0$

$r^6 - r^2 = 0 \iff r^2(r^4 - 1) = 0$

$r^2(r^2-1)(r^2+1) = 0 \iff r \cdot r(r-1)(r+1)(r^2+1) = 0$

$r_1 = 0,\ r_2 = 0,\ r_3 = 1,\ r_4 = -1,\ r_5 = i,\ r_6 = -i$

$y(t) = c_1 + c_2 t + c_3 e^{t} + c_4 e^{-t} + c_5 \sin t + c_6 \cos t$.

34. $4y''' + y' + 5y = 0$

$4r^3 + r + 5 = 0$

$r = -1$ is a root

$(r+1)(4r^2 - 4r + 5) = 0 \Rightarrow r_1 = -1$

$4r^2 - 4r + 5 = 0 \iff (2r-1)^2 + 4 = 0$

$2r - 1 = \pm 2i \Rightarrow r = \frac{1}{2} \pm i$

$y(t) = c_1 e^{-t} + c_2 e^{t/2} \sin t + c_3 e^{t/2} \cos t$.

$$
\begin{array}{r|l}
 & 4r^2 - 4r + 5 \\
\hline
r+1 & 4r^3 + r + 5 \\
 & 4r^3 + 4r^2 \\
\hline
 & -4r^2 + r + 5 \\
 & -4r^2 - 4r \\
\hline
 & +5r + 5 \\
 & 5r + 5 \\
\hline
 &
\end{array}
$$

4.3

5. $y^{(4)} - 4y'' = t^2 + e^t$

$y^{(4)} - 4y'' = 0 \Rightarrow r^4 - 4r^2 = 0 \Leftrightarrow r^2(r^2 - 4) = 0$

$r^2(r-2)(r+2) = 0$

$r_1 = 0,\ r_2 = 0,\ r_3 = -2,\ r_4 = 2$

$y_c(t) = C_1 + C_2 t + C_3 e^{-2t} + C_4 e^{2t}.$

$Y_p(t) = t^2(At^2 + Bt + C) + De^t.$

7. $y^{(6)} + y''' = t$

$r^6 + r^3 = 0 \Leftrightarrow r^3(r^3 + 1) = 0$

$r^3(r+1)(r^2 - r + 1) = 0,\ r = 0, 0, 0, -1, \frac{1}{2} \mp i\frac{\sqrt{3}}{2}$

$y_c(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + C_5 e^{t/2} \cos \frac{\sqrt{3}}{2} t + C_6 e^{\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t$

$Y_p(t) = t^3(At + B)$

11. $y''' - 3y'' + 2y' = t + e^t$

$y''' - 3y'' + 2y' = 0 \Rightarrow r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0 \Leftrightarrow r(r-2)(r-1) = 0 \Rightarrow r = 0, 1, 2$

$y_c(t) = C_1 + C_2 e^t + C_3 e^{2t}$

$Y_p(t) = t(At + B) + Cte^t.$