Ch.2 First Order Diff. Equations

2.1 First Order Linear Diff. Equations

Integrating factor method

The general form of first order diff. eqs is

$$\frac{dy}{dt} = f(t, y)$$

The equation is linear if f is linear in y.

The general form of first order linear diff. eqs is

$$\frac{dy}{dt} + p(t)y = g(t) \qquad (1)$$

first order linear diff. eqs can be solved using the method of integrating factor!

Multiply both sides of the equation

$$y' + p(t)y = g(t)$$

by a certain function $\mu(t)$

$$\mu(t)\,y' + \mu(t)\,p(t)\,y = \mu(t)\,g(t) \qquad -(2)$$

Note that $(\mu(t)y(t))' = \mu y' + \mu' y$. So, if we choose μ such that $\mu'(t) = \mu(t)\,p(t)$ then the left-hand side of equation (2) equals $(\mu y)'$ and the equation becomes

$$(\mu y)' = \mu g$$

which can be solved by direct integration

$$\mu(t)\,y(t) = \int \mu(t)\,g(t)\,dt + c$$

So, the general solution of the diff. eq.

$$y' + p(t)\,y = g(t)$$

is

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\,g(t)\,dt + c\right]$$

The function $\mu(t)$ is called an integrating factor and can be found from the eq.

$$\frac{d\mu}{dt} = \mu(t)\, p(t)$$

$$\int \frac{d\mu}{\mu} = \int p(t)\, dt$$

$$\ln \mu(t) = \int p(t)\, dt$$

$$\Rightarrow \mu(t) = e^{\int p(t)\, dt}$$

Ex. Find the general solution of the diff. eq.

$$y' - y = t^2 e^t \Rightarrow p(t) = -1, \quad g(t) = t^2 e^t$$

$$\mu(t) = e^{-\int dt} = e^{-t}$$

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]$$

$$= \frac{1}{e^{-t}}\left[\int e^{-t}(t^2 e^t)\, dt + c\right]$$

$$= e^t \left[\int t^2\, dt + c\right] = \frac{1}{3} t^3 e^t + c e^t.$$

Ex. Find the Solution of the I.V.P

$$\frac{dy}{dt} - 2y = 4 - t \,, \quad y(0) = y_0$$

$$\mu(t) = e^{-2t}$$

$$y(t) = \frac{1}{e^{-2t}} \left[\int e^{-2t} (4-t)\, dt + c \right]$$

$$\int 4 e^{-2t}\, dt - \int t e^{-2t}\, dt$$

t		e^{-2t}
1	$+$	$\frac{e^{-2t}}{-2}$
0	$-$	$\frac{e^{-2t}}{4}$

$$= -2 e^{-2t} - \left[-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} \right]$$

$$= \frac{1}{2} t e^{-2t} - \frac{7}{4} e^{-2t}$$

$$y(t) = \frac{1}{e^{-2t}} \left[\left(\frac{1}{2} t - \frac{7}{4} \right) e^{-2t} + c \right]$$

$$= \frac{t}{2} - \frac{7}{4} + c e^{2t}$$

$$y(0) = y_0 = -\frac{7}{4} + c \Rightarrow c = y_0 + \frac{7}{4}.$$

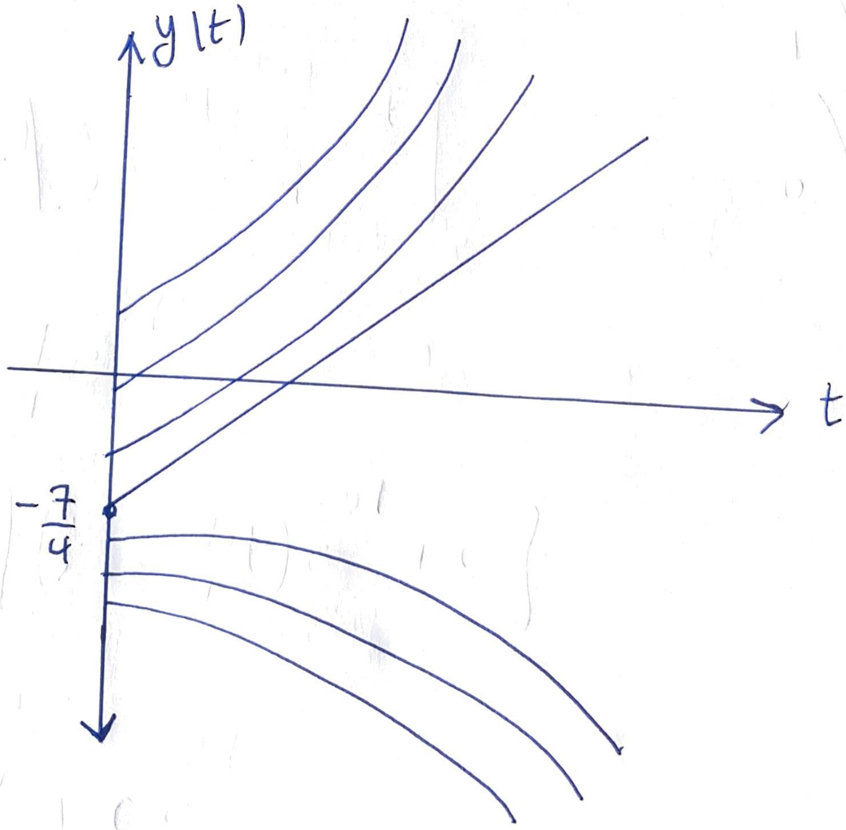
The solution of the IVP is

$$y(t) = \frac{t}{2} - \frac{7}{4} + \left(y_0 + \frac{7}{4}\right) e^{2t}.$$

if $y_0 = -\frac{7}{4} \Rightarrow y(t) = \frac{1}{2}t - \frac{7}{4} \longrightarrow \infty$ as $t \to \infty$

if $y_0 + \frac{7}{4} > 0 \Leftrightarrow y_0 > -\frac{7}{4} \Rightarrow \lim\limits_{t\to\infty} y(t) = +\infty$

if $y_0 + \frac{7}{4} < 0 \Leftrightarrow y_0 < -\frac{7}{4} \Rightarrow \lim\limits_{t\to\infty} y(t) = -\infty.$



Ex. Solve the IVP $ty' + 2y = 4t^2$, $y(1) = y_0$.

$$y' + \frac{2}{t} y = 4t$$

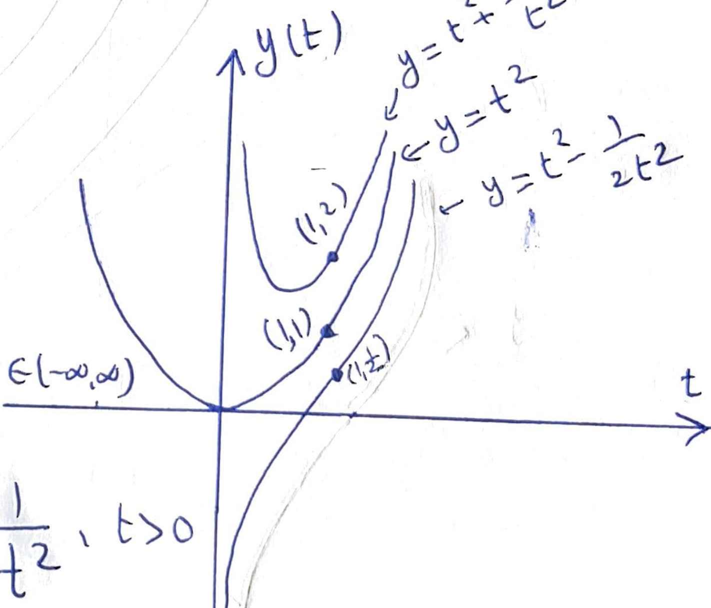
$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$y(t) = \frac{1}{t^2}\left[\int 4t^3 \, dt + c\right] = \frac{1}{t^2}\left[t^4 + c\right]$$

$$y(t) = t^2 + \frac{c}{t^2}$$

$$y(1) = y_0 = 1 + c \Rightarrow c = y_0 - 1$$

$$y(t) = t^2 + \frac{y_0 - 1}{t^2}$$



if $y_0 = 1 \Rightarrow y(t) = t^2$, $t \in (-\infty, \infty)$

if $y_0 = 2 \Rightarrow y(t) = t^2 + \frac{1}{t^2}$, $t > 0$

if $y_0 = \frac{1}{2} \Rightarrow y(t) = t^2 - \frac{1}{2t^2}$, $t > 0$

Solutions of Sec. 2.1

4. $y' + \frac{1}{t}y = 2\sin 2t \, , \quad t > 0$

$$\mu(t) = e^{\int \frac{dt}{t}} = e^{\ln t} = t$$

$$y(t) = \frac{1}{t}\left[\int 2t\sin 2t\,dt + c\right]$$

t	$2\sin 2t$
1 $\xrightarrow{+}$	$-\cos 2t$
0 $\xrightarrow{-}$	$-\frac{\sin 2t}{2}$

$$= \frac{1}{t}\left[-t\cos 2t + \frac{1}{2}\sin 2t + c\right]$$

$$y(t) = -\cos(2t) + \frac{1}{2t}\sin(2t) + \frac{c}{t}$$

15. $ty' + 2y = t^2 - t + 1 \, , \quad y(1) = 2 \, , \quad t > 0$

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$y(t) = \frac{1}{t^2}\left[\int (t^3 - t^2 + t)\,dt + c\right]$$

$$= \frac{1}{t^2}\left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{t^2}{2} + c\right]$$

$$y(1) = \frac{5}{12} + c = 2 \Rightarrow c = \frac{19}{12}$$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{19}{12} = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{25}{12}.$$