2.2 Seperable Equations

A first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

is called seperable if $f(x,y) = \frac{M(x)}{N(y)}$

$$\frac{dy}{dx} = \frac{M(x)}{N(y)} \Leftrightarrow N(y)\,dy = M(x)\,dx$$

$$\Rightarrow \int N(y)\,dy = \int M(x)\,dx.$$

Ex. Solve the diff. eq. $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

$$\Rightarrow \int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C.$$

$$3y - x^3 - y^3 = C.$$

Ex. Solve the IVP $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

$$y(0) = -1 \Rightarrow 3 = C$$

$$y^2 - 2y - (x^3+2x^2+2x+3) = 0$$

This is a quadratic equation in y

$$y(x) = \frac{1}{2}\left[2 \mp \sqrt{4+4(x^3+2x^2+2x+3)}\right]$$

$$= 1 \mp \sqrt{x^3+2x^2+2x+4}$$

Since $y(0) = -1 \Rightarrow y(x) = 1 - \sqrt{x^3+2x^2+2x+4}$.

is the solution of the IVP.

$$x^3+2x^2+2x+4 = x^2(x+2) + 2(x+2) > 0$$

$$(x+2)(x^2+2) > 0 \Leftrightarrow x > -2$$

$$\therefore\ y(x) = 1 - \sqrt{x^3+2x^2+2x+4}\ ,\ x > -2.$$

Solutions Sec. 2.2

2. $y' = \dfrac{3x^2}{y(1+x^3)} \iff \dfrac{dy}{dx} = \dfrac{1}{y} \cdot \dfrac{3x^2}{1+x^3}$

$$\int y\,dy = \int \frac{3x^2}{1+x^3}\,dx$$

$$\frac{1}{2}y^2 = \ln|1+x^3| + c.$$

let $y(0) = \sqrt{2} \Rightarrow 1 = c$

$$y^2 = 2\ln(1+x^3) + 2$$

$$y = \sqrt{2\ln(1+x^3)+2}\ , \quad x > -1.$$

6. $xy' = (1-y^2)^{1/2} \iff \dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x}$$

$$\sin^{-1} y = \ln|x| + C$$

$$y(x) = \sin(\ln|x| + c).$$

20. $y^2(1-x^2)^{1/2}\,dy = \sin^{-1}x\,dx$, $y(0)=1$

$$\int y^2\,dy = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx \Rightarrow \frac{1}{3}y^3 = \frac{1}{2}(\sin^{-1}x)^2 + c$$

$$y(0)=1 \Rightarrow \frac{1}{3} = c \Rightarrow y^3 = \frac{3}{2}(\sin^{-1}x)^2 + 1$$

Ex. Solve the IVP $\frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y}$, $y(0)=1$

Find the interval of definition of $y(x)$.

$$\int (3y^2-6y)\,dy = \int (1+3x^2)\,dx$$

$$y^3 - 3y^2 = x + x^3 + c$$

$$y(0)=1 \Rightarrow -2 = c \Rightarrow y^3 - 3y^2 = x + x^3 - 2$$

$\frac{dy}{dx}$ is undefined when $3y^2-6y = 3y(y-2) = 0 \Rightarrow y = 0, 2$

if $y=0 \Rightarrow x^3 + x - 2 = 0 \Rightarrow x = 1$

if $y=2 \Rightarrow -4 = x^3 + x - 2 \Leftrightarrow x^3 + x + 2 = 0 \Rightarrow x = -1$

$\therefore$ the solution $y(x)$ is defined in the interval

$-1 < x < 1$.

Homogeneous Differential Equations

A differential equation $\frac{dy}{dx} = f(x,y)$ is called homogeneous if $f(x,y)$ can be expressed as a function of the ratio $\frac{y}{x}$. Such equations can be solved by substituting $v(x) = \frac{y}{x}$

$$\Rightarrow y(x) = x v(x)$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

The diff. eq. becomes:

$$v + x\frac{dv}{dx} = f(1,v)$$

$$x\frac{dv}{dx} = f(1,v) - v$$

This is a seperable differential equation:

$$\int \frac{dv}{f(1,v)-v} = \int \frac{dx}{x}$$

30. Solve the eq. $\frac{dy}{dx} = \frac{y-4x}{x-y}$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}, \quad \text{let } v = \frac{y}{x} \Rightarrow y = x\,v(x)$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v \Leftrightarrow x\frac{dv}{dx} = \frac{\cancel{v} - 4 + v^2 - \cancel{v}}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2-4}{1-v} \Leftrightarrow \frac{1-v}{v^2-4}\,dv = \frac{dx}{x}$$

$$\frac{1-v}{v^2-4} = \frac{1-v}{(v-2)(v+2)} = \frac{A}{v-2} + \frac{B}{v+2}$$

$$1 - v = A(v+2) + B(v-2)$$

$$v = 2 \Rightarrow -1 = 4A \Rightarrow A = -\frac{1}{4}$$

$$v = -2 \Rightarrow 3 = -4B \Rightarrow B = -\frac{3}{4}$$

$$\frac{1-v}{v^2-4} = \frac{1/4}{v-2} - \frac{3/4}{v+2}$$

$$\int \frac{1-v}{v^2-4}\,dv = \frac{1}{4}\int \frac{dv}{v-2} - \frac{3}{4}\int \frac{dv}{v+2}$$

$$\frac{1}{4}\int\frac{dv}{v-2}-\frac{3}{4}\int\frac{dv}{v+2}=\int\frac{dx}{x}$$

$$\frac{1}{4}\ln|v-2|-\frac{3}{4}\ln|v+2|=\ln|x|+C$$

$$\frac{1}{4}\ln\left|\frac{y}{x}-2\right|-\frac{3}{4}\ln\left|\frac{y}{x}+2\right|=\ln|x|+C$$

31. $$\frac{dy}{dx}=\frac{2x^2+xy+y^2}{x^2}$$

$$\frac{dy}{dx}=2+\frac{y}{x}+\left(\frac{y}{x}\right)^2 .$$ Let $v=\frac{y}{x}$

$$\cancel{v}+x\frac{dv}{dx}=2+\cancel{v}+v^2 \Leftrightarrow x\frac{dv}{dx}=2+v^2$$

$$\int\frac{dv}{2+v^2}=\int\frac{dx}{x}$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right)=\ln|x|+C$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}x}\right)=\ln|x|+C.$$