2.4 Differences Between Linear and Nonlinear Eqs.

Existence and Uniqueness of Solutions

I. Linear Equations

Theorem: If $p(t)$ and $g(t)$ are continuous function on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the diff. eq.

$$y' + p(t)y = g(t)$$

for all $t \in I$, and that also satisfies the I.C.

$$y(t_0) = y_0$$

where y_0 is an arbitrary initial value.

Rem. In this case $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\,dt + c\right]$

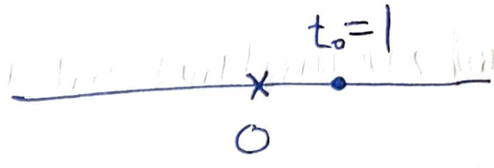
$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\,ds + y_0\right], \quad \mu(t) = e^{\int_{t_0}^{t} p(s)\,ds}$$

Ex. Find an interval in which the IVP

$$ty' + 2y = 4t^2 \,, \quad y(1) = 2$$

has a unique solution.

$$y' + \frac{2}{t} y = 4t$$



$$p(t) = \frac{2}{t} \,, \quad g(t) = 4t$$

$p(t)$ is Continuous on $(-\infty, 0) \cup (0, \infty)$

$g(t)$ is Continuous on $(-\infty, \infty)$

$$I = (0, \infty).$$

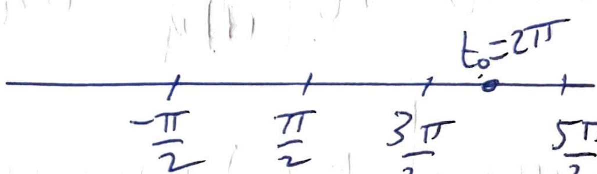
3. $y' + \frac{\sin t}{\cos t} y = \sin t \,, \quad y(2\pi) = 0$

$p(t) = \frac{\sin t}{\cos t}$ Conts on $(-\infty, \infty) \setminus \{ \frac{\pi}{2} + n\pi \}$

$n = 0, \pm 1, \pm 2, --$

$g(t) = \sin t$ Conts on $(-\infty, \infty)$

$$I = \left(\frac{3\pi}{2}, \frac{5\pi}{2} \right).$$



II Nonlinear Equations

Consider the diff. eq. $\frac{dy}{dt} = f(t,y)$

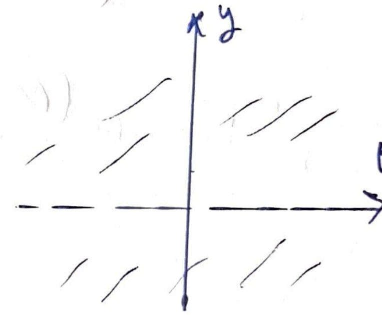
$\Rightarrow$ f is defined at point in the ty-plane

we can differentiate f with respect to t and y.

ex. $f(t,y) = ty$ defined at all points in the ty-plane

$f_y = t$ or $\frac{\partial f}{\partial y} = t$

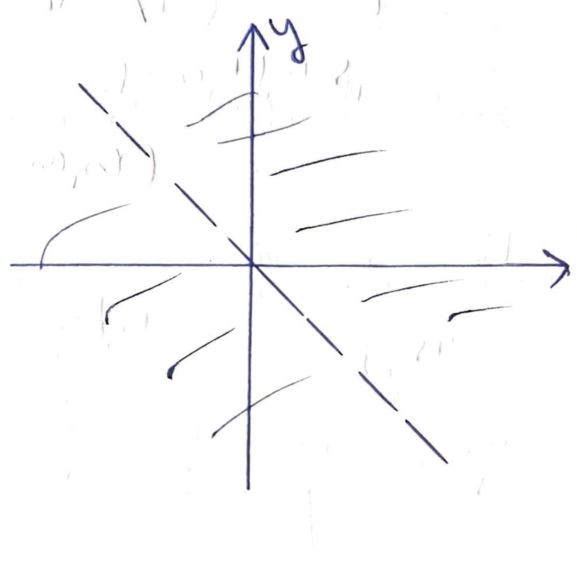
ex. $f(t,y) = \frac{t}{y} \Rightarrow y \neq 0$



$\frac{\partial f}{\partial y} = f_y(t,y) = -\frac{t}{y^2}$, $y \neq 0$

ex. $f(t,y) = \frac{1}{t+y}$, $t+y \neq 0 \Rightarrow y \neq -t$

$\frac{\partial f}{\partial y} = f_y(t,y) = -\frac{1}{(t+y)^2}$

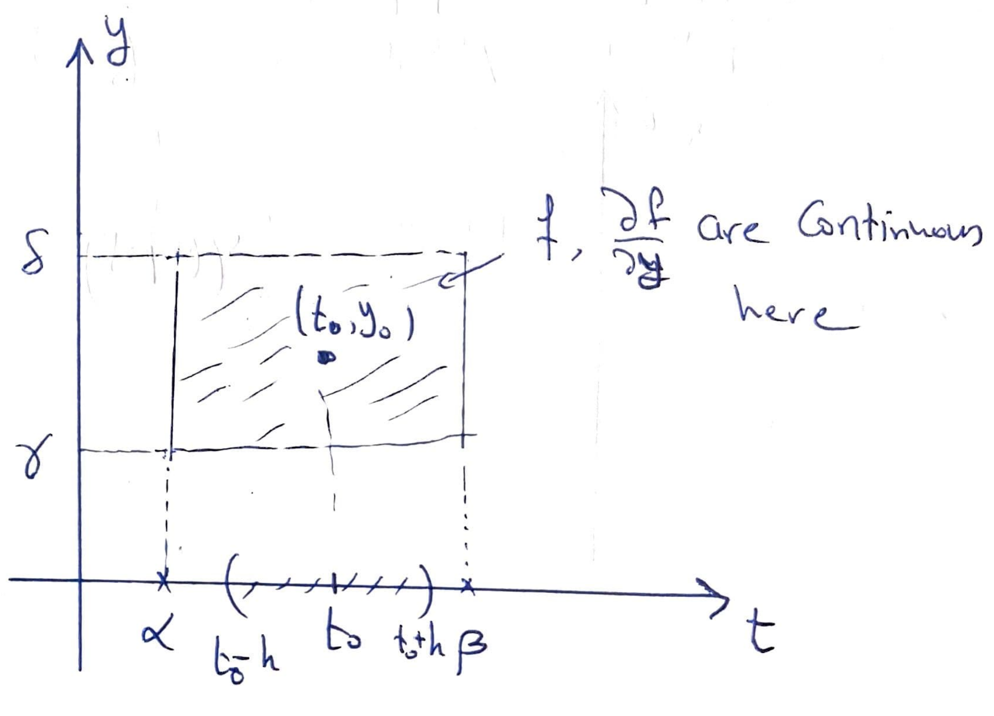


Theorem. Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle:

$R: \alpha < t < \beta, \ \gamma < y < \delta$

containing the point (t_0, y_0).

Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem: $y' = f(t, y), \ y(t_0) = y_0$.



Ex. Apply the previous theorem to the IVP

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$

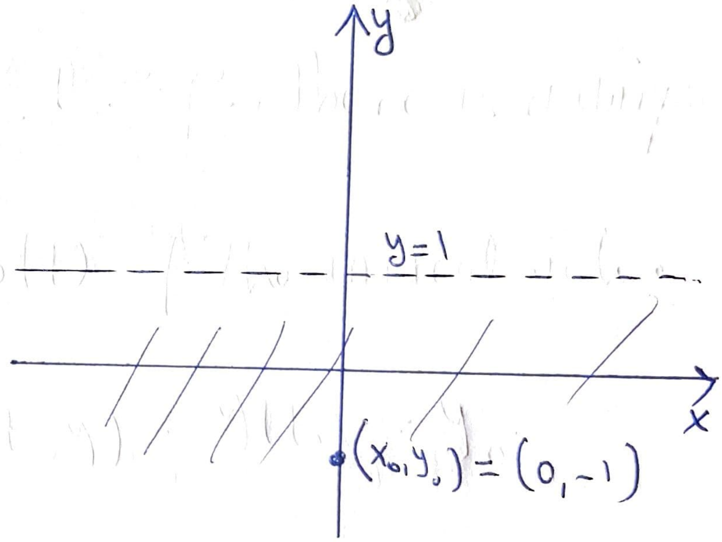
$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)}$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

Continuous everywhere except on the line $y=1$

$R: -\infty < x < \infty$

$-\infty < y < 1$



from Sec. 2.2.

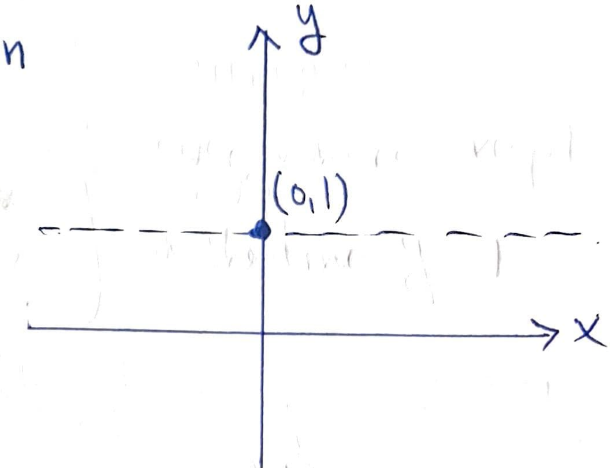
The IVP has the unique solution

$$y(x) = 1 - \sqrt{x^3+2x^2+2x+4}$$

defined on the interval $(-2,\infty)$.

Now, Suppose that we change the initial Condition to $y(0)=1$

No rectangle can be drawn about $(0,1)$ in which $f, \frac{\partial f}{\partial y}$ are Continuous



Solving the IVP we get

$$y^2-2y = x^3+2x^2+2x+C$$

$$y(0)=1 \Rightarrow C=-1$$

$$y^2-2y-(x^3+2x^2+2x-1)=0$$

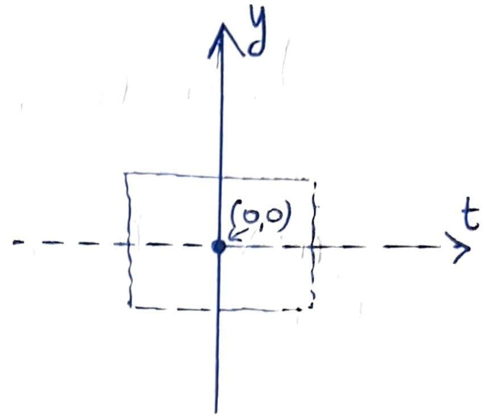
Solving the quadratic equation, we get

$$y(x) = 1 \pm \sqrt{x^3+2x^2+2x}$$

So, we get 2 functions that satisfy the diff. eq. for $x>0$.

Ex. Consider the IVP

$$y' = y^{1/3} \quad , \ y(0) = 0$$



$$f(t,y) = y^{1/3}$$

$$f_y(t,y) = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$$

f, f_y are Continuous for $y \neq 0$

the conditions of the existence and uniqueness theorem are not satisfied. There is no rectangle R Containing $(0,0)$ in which f, f_y are Continuous.

Solving this seperable equation:

$$\frac{dy}{dt} = y^{1/3}$$

$$\int y^{-1/3}\, dy = \int dt$$

$$\frac{3}{2} y^{2/3} = t + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$y^{2/3} = \frac{2}{3} t \quad , \ t \geq 0$$

$$y^2 = \left(\frac{2}{3}t\right)^3$$

$$y = \pm\left(\frac{2}{3}t\right)^{3/2}$$

$\Rightarrow$ $y = \phi_1(t) = \left(\frac{2}{3}t\right)^{3/2}$ is a first solution.

$y = \phi_2(t) = -\left(\frac{2}{3}t\right)^{3/2}$ is a second solution

of the IVP.

In fact, this IVP has infinitely many solutions

for any t_0, the following functions are

solutions of the IVP

$$y = \begin{cases} 0 , & 0 \le t < t_0 \\ \pm\left[\frac{2}{3}(t-t_0)\right]^{3/2} , & t \ge 0 \end{cases}$$

~Sec. 2.4

This example shows how the interval of definition depends on the I.C., precisely on y_0.

Example: $y' = y^2$, $y(0) = y_0$ $\Rightarrow f(t,y) = y^2$

Solving this seperable equation,

$$\int y^{-2}\,dy = \int dt$$

$$-y^{-1} = t + c$$

$$y(0) = y_0 \Rightarrow -\frac{1}{y_0} = c$$

$$-y^{-1} = t - \frac{1}{y_0} \Rightarrow \frac{1}{y} = \frac{1 - y_0 t}{y_0}$$

$$y(t) = \frac{y_0}{1 - y_0 t} \Rightarrow t \neq \frac{1}{y_0}$$

f, $f_y = 2y$ are continuous at all points. the thm is satisfied at all points of the ty-plane.

if $y_0 > 0 \Rightarrow I = \left(-\infty, \frac{1}{y_0}\right)$

$t_0 = 0$ $\quad \frac{1}{y_0} > 0$

if $y_0 < 0 \Rightarrow I = \left(\frac{1}{y_0}, \infty\right)$

$\frac{1}{y_0} < 0$ $\quad 0 = t_0$

we choose the interval to which $t_0 = 0$ belongs.

14. $y' = 2ty^2, \quad y(0) = y_0$. Find the interval I.

$$\int y^{-2}\,dy = \int 2t\,dt$$

$$-\frac{1}{y} = t^2 + c$$

$$y(0) = y_0 \Rightarrow -\frac{1}{y_0} = c.$$

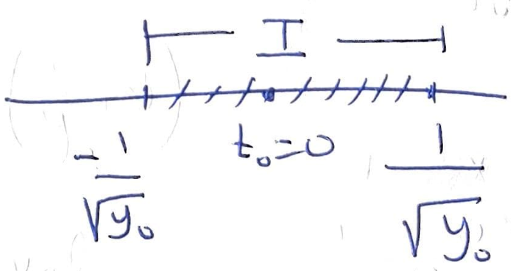
$$-\frac{1}{y(t)} = t^2 - \frac{1}{y_0} \Rightarrow \frac{1}{y(t)} = \frac{1 - y_0 t^2}{y_0}$$

$$y(t) = \frac{y_0}{1 - y_0 t^2}$$

if $y_0 < 0 \Rightarrow 1 - y_0 t^2 \neq 0 \Rightarrow I = (-\infty, \infty)$

if $y_0 > 0 \Rightarrow 1 - y_0 t^2 = 0 \Leftrightarrow t = \pm \frac{1}{\sqrt{y_0}}$

$$\Rightarrow I = \left(-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}\right)$$



Bernoulli Equations, P77

Bernoulli equations are equations of the form

$y' + p(t)\,y = q(t)\,y^n$

if $n = 0, 1 \Rightarrow$ this eq. is linear (sec. 2.1)

So, we assume that $n = 2, 3, 4, \cdots$

Dividing both sides by y^n, we get

$y^{-n} y' + p(t)\, y^{1-n} = q(t)$

Let $v(t) = y^{1-n} \Rightarrow v'(t) = (1-n)\, y^{-n} y'$

substitute into the eq., we get

$\frac{1}{1-n} v'(t) + p(t)\, v(t) = q(t)$.

$v'(t) + (1-n)\, p(t)\, v(t) = (1-n)\, q(t)$. — $(*)$

28. $t^2 y' + 2ty = y^3, \quad t > 0$

$n = 3$, divide by $t^2 \Rightarrow y' + \frac{2}{t} y = \frac{1}{t^2} y^3$

$p(t) = \frac{2}{t}, \quad q(t) = \frac{1}{t^2}$.

from $(*)$: $v'(t) - \frac{4}{t} v(t) = \frac{-2}{t^2}$, $\quad v(t) = \frac{1}{y^2}$

this is a first order linear equation.

the integrating factor is

$$\mu(t) = e^{-4\int \frac{dt}{t}} = e^{-4\ln t} = t^{-4}$$

$$v(t) = \frac{1}{t^{-4}}\left[\int t^{-4}\left(\frac{-2}{t^2}\right)dt + c\right]$$

$$= \frac{1}{t^{-4}}\left[-2\int t^{-6}\,dt + c\right]$$

$$= t^{4}\left[\frac{-2t^{-5}}{-5} + c\right]$$

$$= \frac{2}{5}t^{-1} + c\,t^{4}$$

$$v(t) = \frac{2 + c_1 t^5}{5} \quad \text{where } c_1 = 5c$$

but $v(t) = \frac{1}{y^2}$

$$\frac{1}{y^2} = \frac{2 + c_1 t^5}{5}$$

$$y^2 = \frac{5}{2 + c_1 t^5}$$

$$y(t) = \pm\left[\frac{5}{2 + c_1 t^5}\right]^{\frac{1}{2}}$$