2.6 Exact Equations and Integrating Factors

Review of Partial Derivatives:

Consider a function $\psi(x,y) = x^3 y^2$

the first order partial derivatives of ψ are

$$\psi_x = \frac{\partial \psi}{\partial x} = 3x^2 y^2$$

$$\psi_y = \frac{\partial \psi}{\partial y} = 2x^3 y$$

Second order partial derivatives

$$\left.\begin{aligned} \psi_{xx} &= 6xy^2, \quad \psi_{xy} = 6x^2 y \\ \psi_{yy} &= 2x^3, \quad \psi_{yx} = 6x^2 y \end{aligned}\right\} \psi_{xy} = \psi_{yx}.$$

Ex. $\psi(x,y) = \sin(xy)$

$\psi_x = y\cos(xy)$, $\psi_{xy} = \cos(xy) - xy\sin(xy)$

$\psi_y = x\cos(xy)$, $\psi_{yx} = \cos(xy) - xy\sin(xy)$

$\psi_{xy} = \psi_{yx}$.

Defn A differential equation $M(x,y)+N(x,y)y'=0$

is called exact if there is a function

$\psi(x,y)$ such that:

$$\psi_x(x,y)=M(x,y),$$
$$\psi_y(x,y)=N(x,y)$$

If the equation $M(x,y)+N(x,y)\frac{dy}{dx}=0$

is exact, then, it can be written as

$$\psi_x(x,y)+\psi_y(x,y)\frac{dy}{dx}=0$$

$$\Leftrightarrow \frac{d}{dx}\psi(x,y(x))=0$$

$$\Rightarrow \psi(x,y)=c$$

This equation gives the solution $y(x)$

implicitly.

Test ~~for~~ exactness:

if $\Psi_x = M \Rightarrow \Psi_{xy} = M_y$

$\Psi_y = N \Rightarrow \Psi_{yx} = N_x$ $\Rightarrow M_y = N_x$

Theorem. Let the functions M, N, M_y and N_x be continuous in the rectangular region

$R: \alpha < x < \beta, \gamma < y < \delta$

Then, the equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is an exact differential equation if and only if

$$M_y(x,y) = N_x(x,y)$$

for all $(x,y) \in R$.

Example. Solve the diff eq.

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\,y' = 0$$

$$M(x,y) = y\cos x + 2xe^y$$

$$N(x,y) = \sin x + x^2e^y - 1$$

$$M_y = \cos x + 2xe^y$$

$$N_x = \cos x + 2xe^y$$

$M_y = N_x \Rightarrow$ the eq. is exact.

$$\psi_x = M = y\cos x + 2xe^y$$

Integrating with respect to x, we get

$$\psi(x,y) = y\sin x + x^2e^y + h(y)$$

To find $h(y)$, we use the Condition $\psi_y = N$

$$\Rightarrow \psi_y = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1$$

$$\Rightarrow h'(y) = -1 \Rightarrow h(y) = -y$$

The solution $y(x)$ satisfies the implicit equation

$$\psi(x,y) = y\sin x + x^2e^y - y = c$$

Ex. Solve the IVP

$$\left(\frac{y}{x}+4x\right)dx+(\ln x-1)\,dy=0, \quad y(1)=1, \ x>0$$

$$M=\frac{y}{x}+4x, \quad N=\ln x-1$$

$$M_y=\frac{1}{x}=N_x \Rightarrow \text{exact}$$

$$\psi_x=M=\frac{y}{x}+4x$$

$$\psi(x,y)=\int\left(\frac{y}{x}+4x\right)dx+h(y)$$

$$=y\ln x+2x^2+h(y)$$

$$\psi_y=\ln x+h'(y)=N=\ln x-1$$

$$\Rightarrow h'(y)=-1 \Rightarrow h(y)=-y$$

$$\psi(x,y)=y\ln x+2x^2-y=c$$

$$y(1)=1 \Rightarrow 0+2-1=c \Rightarrow c=1$$

$$y\ln x+2x^2-y=1$$

$$y(\ln x-1)=1-2x^2$$

$$y(x)=\frac{1-2x^2}{\ln x-1}, \quad x\in(0,e)$$

Integrating factors:

In some cases, when the equation $M + N y' = 0$

is not exact, we can find a function μ

such that the equation $\mu M + \mu N y' = 0$ is exact.

We consider two cases:

Case 1: μ is a function of x only: $\mu(x)$

If the ratio $\frac{M_y - N_x}{N} = Q(x)$ is a function

of x only $\Rightarrow \exists\ \mu(x)$ and $\mu(x) = e^{\int Q(x)\,dx}$.

Example: Solve the eq. $(3xy + y^2) + (x^2 + xy)\, y' = 0$

$M = 3xy + y^2$

$N = x^2 + xy$

$M_y = 3x + 2y$ $\Rightarrow M_y \neq N_x \Rightarrow$ Not exact

$N_x = 2x + y$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy}$$

$$= \frac{x + y}{x(x+y)} = \frac{1}{x}$$

$$\Rightarrow \varphi(x) = \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x.$$

Multiply the eq. by x,

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

$$M = 3x^2y + xy^2$$

$$N = x^3 + x^2y$$

$$M_y = 3x^2 + 2xy \quad \Rightarrow M_y = N_x \Rightarrow \text{exact}$$

$$N_x = 3x^2 + 2xy$$

$$\psi_x = M = 3x^2y + xy^2$$

$$\Rightarrow \psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(y)$$

$$\psi_y = x^3 + x^2y + h'(y) = N = x^3 + x^2y$$

$$h'(y) = 0 \Rightarrow h(y) = 0$$

$$\therefore \psi(x,y) = x^3y + \frac{1}{2}x^2y^2 = c.$$

Case: μ is a function of y only, $\mu(y)$

There is an integrating factor $\mu(y)$ if

$Q(y) = \dfrac{N_x - M_y}{M}$ is a function of y only.

$$\mu(y) = e^{\int Q(y)\,dy}.$$

example (20). $y\,dx + (2x - ye^y)\,dy = 0$

$$M = y, \quad N = 2x - ye^y$$

$$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}$$

$$\mu(y) = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

Multiply the eq. by y: $y^2\,dx + (2xy - y^2e^y)\,dy = 0$

$$M = y^2, \quad N = 2xy - y^2e^y$$

$$M_y = 2y = N_x$$

$$\psi_x = M = y^2 \Rightarrow \psi(x,y) = xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = N = 2xy - y^2e^y$$

$$\Rightarrow h'(y) = -y^2e^y$$

$$h(y) = -\int y^2 e^y \, dy$$

integrating by parts :

$$\begin{array}{cc} y^2 & e^y \\ 2y & e^y \\ 2 & e^y \\ 0 & e^y \end{array}$$

$$h(y) = -\left[y^2 e^y - 2ye^y + 2e^y\right]$$

$$\Psi(x,y) = xy^2 - y^2 e^y + 2ye^y - 2e^y = c.$$

Ex. Solve the IVP

$$(xy-1)\,dx + x^2\,dy = 0 \quad , \quad y(1) = 1$$

$$M = xy-1 \quad , \quad N = x^2$$

$$M_y = x \neq N_x = 2x$$

$$\frac{M_y - N_x}{N} = \frac{x-2x}{x^2} = \frac{-x}{x^2} = -\frac{1}{x}$$

$$\Rightarrow \exists\ \mu(x) = e^{\int -\frac{dx}{x}} = e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Multiply the eq. by $\frac{1}{x}$

$$\left(y - \frac{1}{x}\right) dx + x\,dy = 0$$

$$M = y - \frac{1}{x} \quad , \quad N = x$$

$$\Rightarrow M_y = 1 = N_x \Rightarrow \text{exact.}$$

$$\psi_x = y - \frac{1}{x} \Rightarrow \psi(x,y) = \int \left(y - \frac{1}{x}\right) dx + h(y)$$

$$\psi(x,y) = xy - \ln x + h(y)$$

$$\psi_y = x - 0 + h'(y) = x = N$$

$$h'(y) = 0 \Rightarrow h(y) = 0$$

$$\psi(x,y) = xy - \ln x = c$$

$$y(1) = 1 \Rightarrow 1 = c \Rightarrow xy - \ln x = 1$$

$$\Rightarrow y(x) = \frac{1+\ln x}{x} \quad , \quad x > 0 .$$

Ex. Solve the diff. eq. $y\,dx - (2x+y^4)\,dy = 0$

$$M = y \Rightarrow M_y = 1$$
$$N = -2x - y^4 \Rightarrow N_x = -2 \quad \Rightarrow M_y \neq N_x \Rightarrow \text{not exact.}$$

$$\frac{M_y - N_x}{N} = \frac{1+2}{-(2x+y^4)} = \frac{-3}{2x+y^4} \neq Q(x) \Rightarrow \nexists\, \mu(x)$$

$$\frac{N_x - M_y}{M} = \frac{-2-1}{y} = \frac{-3}{y} = Q(y) \Rightarrow \exists\, \mu(y)$$

$$\mu(y) = e^{-3\int \frac{dy}{y}} = e^{-3\ln y} = y^{-3} = \frac{1}{y^3}$$

Multiply the eq. by $\mu(y) = \frac{1}{y^3}$

$$\frac{1}{y^2}\,dx - (2xy^{-3} + y)\,dy = 0$$

$$M = \frac{1}{y^2} \Rightarrow M_y = \frac{-2}{y^3} \quad \Rightarrow M_y = N_x \Rightarrow \text{exact}$$
$$N = -2xy^{-3} - y \Rightarrow N_x = -\frac{2}{y^3}$$

$$\psi(x,y) = \int \frac{1}{y^2}\,dx + h(y) = \frac{x}{y^2} + h(y)$$

$$\psi_y = \cancel{-\frac{2x}{y^3}} + h'(y) = N = \cancel{-\frac{2x}{y^3}} - y$$

$$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2$$

$$\psi(x,y) = \frac{x}{y^2} - \frac{1}{2}y^2 = c.$$