2.8 The Existence & Uniqueness Theorem

Consider the IVP $y' = f(t,y)$, $y(0) = 0$

According the existence and uniqueness theorem in sec. 2.4, if f, f_y are continuous in a rectangle containing $(0,0)$, then the IVP has a unique solution.

If $y = \phi(t)$ is a solution of the IVP, then

$$\phi'(t) = f(t, \phi(t))$$

integrating, we get the integral equation

$$\phi(t) = \int_0^t f(s, \phi(s))\, ds$$

note that $\phi(0) = 0$.

any solution of the integral equation is a solution of the diff. eq. and conversely.

One method for solving the integral equation is the method of successive approximations or Picard's iteration method.

we start be choosing an initial function $\phi_0(t) = 0$, then we define a sequence of functions by

$$\phi_{k+1}(t) = \int_0^t f(s, \phi_k(s))\, ds \;, \quad k = 0, 1, 2, \ldots$$

The terms of the sequence $\phi_0, \phi_1, \ldots$ satisfy the ICs but not the diff. eq.

The limit of the sequence is the solution of the IVP: $\phi(t) = \lim_{k \to \infty} \phi_k(t)$

Example. Use Picard's method to solve the IVP

$$y' = 2t(1+y) \;, \quad y(0) = 0 .$$

in this example $f(t, y) = 2t(1+y)$

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t 2s(1 + \phi_0(s))\, ds$$

$$= \int_0^t 2s\, ds = s^2 \Big|_0^t = t^2 .$$

$$\phi_2(t) = \int_0^t 2s\,(1+\phi_1(s))\,ds$$

$$= \int_0^t 2s(1+s^2)\,ds = \int_0^t (2s+2s^3)\,ds$$

$$= s^2 + \frac{s^4}{2!}\Big|_0^t = t^2 + \frac{t^4}{2!}$$

$$\phi_3(t) = \int_0^t 2s(1+\phi_2(s))\,ds$$

$$= \int_0^t 2s\left(1+s^2+\frac{s^4}{2!}\right)ds$$

$$= \int_0^t \left(2s+2s^3+2\frac{s^5}{2!}\right)ds$$

$$= s^2+\frac{s^4}{2!}+\frac{s^6}{3!}\Big|_0^t$$

$$= t^2+\frac{t^4}{2!}+\frac{t^6}{3!}$$

We conclude that $\phi_n(t) = t^2+\frac{t^4}{2!}+\cdots+\frac{t^{2n}}{n!}$

$$= \sum_{k=1}^{n} \frac{t^{2k}}{k!}$$

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$$

note that $e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow \phi(t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} - 1$$

$$= \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!} - 1$$

$$= e^{t^2} - 1$$

4. Solve the IVP $y' = -y - 1$, $y(0) = 0$. using Picard's iteration method.

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t (-\phi_0(s) - 1)\, ds$$

$$= -\int_0^t ds$$

$$= -s \Big|_0^t$$

$$= -t$$

$$\phi_2(t) = \int_0^t \left(-\phi_1(s) - 1\right) ds$$

$$= \int_0^t (s-1)\, ds = \frac{s^2}{2} - s \Big|_0^t = \frac{t^2}{2} - t$$

$$\phi_3(t) = \int_0^t \left(-\phi_2(s) - 1\right) ds$$

$$= \int_0^t \left(-\frac{s^2}{2} + s - 1\right) ds$$

$$= -\frac{s^3}{3!} + \frac{s^2}{2!} - s \Big|_0^t$$

$$= -\frac{t^3}{3!} + \frac{t^2}{2!} - t$$

$$\vdots$$

$$\phi_n(t) = -t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots + (-1)^n \frac{t^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-t)^n}{n!}$$

$$= e^{-t} - 1.$$