7.5 Homogeneous Linear Systems with Constant Coefficients

We consider systems of diff. eqs of the form

$\vec{x}' = A\vec{x}$ where A is $n \times n$ matrix

when $n=2$, this system is given by

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\vec{x}' = A\vec{x} \iff \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1' = a x_1 + b x_2$$

$$x_2' = c x_1 + d x_2$$

· A solution of the system has the form

$$\vec{x} = \vec{v} e^{rt}$$

for some vector $\vec{v}$ and scalar r.

How to find $\vec{v}$ and r ??

$\vec{x}(t) = \vec{v}e^{rt}, \quad \vec{x}'(t) = r\vec{v}e^{rt}$

Substitute into the system $\vec{x}' = A\vec{x}$

$r\vec{v}e^{rt} = A\vec{v}e^{rt}$

$\Leftrightarrow \quad A\vec{v} = r\vec{v} \quad \Leftrightarrow A\vec{v} - r\vec{v} = 0$

$\Leftrightarrow \quad (A - rI)\vec{v} = 0, \quad \vec{v} \neq \vec{0},$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

The system $(A-rI)\vec{v} = \vec{0}$ has nonzero

solution $\vec{v} \Leftrightarrow \det(A - rI) = 0$

r is called an eigenvalue of A

$\vec{v}$ is an eigenvector associated with

the eigenvalue r.

Rem. $\det(A-rI)$ is a polynomial of degree 2

if A is a 2×2 matrix with roots r_1, r_2

(1) r_1, r_2 real, (2) r_1, r_2 Complex, (3) $r_1 = r_2$ repeated.

Ex Solve the system $\vec{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{X}$

We must find the eigenvalues and corresponding eigenvectors of A.

$$\det(A - rI) = \det\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}$$

$$= (1-r)^2 - 4 = 0$$

$$= r^2 - 2r - 3 = 0$$

$$\therefore \det(A - rI) = (r-3)(r+1) = 0$$

$$\Rightarrow r_1 = 3, \quad r_2 = -1$$

Eigenvectors:

$r_1 = 3$: $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-2v_1 + v_2 = 0,$$

$$4v_1 - 2v_2 = 0$$

$$\Leftrightarrow v_2 = 2v_1 \Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$r = -1$: $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2v_1 + v_2 = 0$

$4v_1 + 2v_2 = 0$

$v_2 = -2v_1 \Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\vec{v}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

we get two solutions!

$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$, $\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$$W(x^{(1)}, x^{(2)}) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t}$$

$\therefore$ $\vec{x}^{(1)}(t)$, $\vec{x}^{(2)}(t)$ form a F.S. of solutions

general solution:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

2. $\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \vec{x}$

$$\det\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} = (1-r)(-4-r)+6$$

$$= r^2+3r+2$$

$$= (r+1)(r+2) = 0 \Rightarrow r_1 = -1,\ r_2 = -2$$

Eigenvectors: $r_1 = -1$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 2v_1 - 2v_2 = 0$$

$$\Leftrightarrow v_1 = v_2$$

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$r = -2$:
$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 3v_1 - 2v_2 = 0$$

$$\Leftrightarrow v_2 = \frac{3}{2} v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ \frac{3}{2} v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

$$\vec{v}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\quad \vec{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

7.6 Complex Eigenvalues

$$r_1 = \lambda + i\mu \ , \ r_2 = \lambda - i\mu$$

$$\vec{v}^{(1)} = \vec{a} + i\vec{b} \ , \ \vec{v}^{(2)} = \vec{a} - i\vec{b}$$

$$\vec{x}(t) = \vec{v}e^{rt} = (\vec{a} + i\vec{b})e^{(\lambda + i\mu)t}$$

$$= (\vec{a} + i\vec{b})\, e^{\lambda t} e^{i\mu t}$$

$$= (\vec{a} + i\vec{b})\left[e^{\lambda t}\cos\mu t + i e^{\lambda t}\sin\mu t\right]$$

$$= \left(\vec{a} e^{\lambda t}\cos\mu t - \vec{b} e^{\lambda t}\sin\mu t\right)$$

$$+ i\left(\vec{a} e^{\lambda t}\sin\mu t + \vec{b} e^{\lambda t}\cos\mu t\right)$$

We get the real solutions

$$\vec{u}(t) = e^{\lambda t}\left(\vec{a}\cos\mu t - \vec{b}\sin\mu t\right)$$

$$\vec{v}(t) = e^{\lambda t}\left(\vec{a}\sin\mu t + \vec{b}\cos\mu t\right).$$

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

Ex. Solve the system $\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}$

$$\det(A - rI) = \begin{vmatrix} -\frac{1}{2}-r & 1 \\ -1 & -\frac{1}{2}-r \end{vmatrix}$$

$$= \left(\frac{1}{2}+r\right)^2 + 1 = 0$$

$$\Leftrightarrow \quad r = -\frac{1}{2} \mp i \,, \quad \lambda = -\frac{1}{2}, \quad \mu = 1$$

Eigenvectors:

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -iv_1 + v_2 = 0$$

$$\Leftrightarrow v_2 = iv_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{u}(t) = e^{\lambda t}\left(\vec{a} \cos \mu t - \vec{b} \sin \mu t\right)$$

$$= e^{-t/2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t\right)$$

$$= e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\vec{v}(t) = e^{\lambda t}\left[a \sin \mu t + b \cos \mu t\right]$$

$$= e^{-t/2}\left[\begin{pmatrix}1\\0\end{pmatrix}\sin t + \begin{pmatrix}0\\1\end{pmatrix}\cos t\right]$$

$$= e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

general solution

$$\vec{X}(t) = c_1 e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

Consider the I.C. $\vec{X}(0) = \begin{pmatrix}1\\-1\end{pmatrix}$

$$\vec{X}(0) = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \Leftrightarrow \begin{matrix}c_1 = 1\\ c_2 = -1\end{matrix}$$

$$\vec{X}(t) = e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} - e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

$$\vec{x}(t) = e^{-t/2}\begin{pmatrix}\cos t - \sin t\\ -\sin t - \cos t\end{pmatrix}$$

(2) $\vec{X}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{X}$

$$\det\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} = (2-r)(-2-r)+5$$

$$= r^2+1 = 0 \Rightarrow r = \pm i, \quad \lambda = 0, \mu = 1$$

Vectors: $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$v_1 - (2+i)v_2 = 0 \Leftrightarrow v_1 = (2+i)v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (2+i)v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\vec{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{u}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$$

$$\vec{v}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

7.8 Repeated Eigenvalues

If r is a repeated eigenvalue of A and $\vec{v}$ is the corresponding eigenvector, then

$$\vec{x}^{(1)}(t) = \vec{v}e^{rt}$$

A second solution $x^{(2)}(t)$ is given by

$$\vec{x}^{(2)}(t) = \vec{v}te^{rt} + \vec{w}e^{rt}$$

where w is a solution of the system

$$(A - rI)\vec{w} = \vec{v}.$$

The general solution is given by

$$\vec{x}(t) = c_1\vec{v}e^{rt} + c_2\left(\vec{v}te^{rt} + \vec{w}e^{rt}\right)$$

Ex. Consider the system $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$

$$\det \begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} = (1-r)(3-r)+1$$
$$= r^2 - 4r + 4$$
$$= (r-2)^2 = 0$$

$\Rightarrow r_1 = r_2 = 2.$

Eigenvector: $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow v_1 + v_2 = 0 \Leftrightarrow v_2 = -v_1$

$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$

To find a second solution, we solve

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$-w_1 - w_2 = 1$

$\Leftrightarrow w_1 + w_2 = -1$

$$w_2 = -1 - w_1$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ -1-w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\vec{x}^{(2)}(t) = \vec{v}\, t e^{rt} + \vec{w} e^{rt}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \Bigg(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + w_1 \underbrace{\cancel{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \Bigg) e^{2t}}_{x^{(1)}(t)}$$

$$\vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$$

8. $\vec{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

$$\det(A - rI) = \begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = (1-r)(-7-r) + 16$$

$$= r^2 + 6r + 9$$

$$= (r+3)^2 = 0$$

$$r_1 = r_2 = -3$$

Eigenvectors:

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow 4v_1 - 4v_2 = 0 \Leftrightarrow v_1 = v_2$$

$$\vec{v}^{(1)} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad x^{(1)}(t) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} e^{-3t}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$4w_1 - 4w_2 = 4 \Leftrightarrow w_2 = -1 + w_1$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ -1 + w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \vec{x}^{(2)}(t) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-3t}$$

$$\vec{X}(t) = c_1 \begin{pmatrix} 4 \\ 4 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-3t} \right]$$

$$\text{I.C.} \Rightarrow c_1 \begin{pmatrix} 4 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$4c_1 = 3 \Rightarrow c_1 = 3/4$$

$$4c_1 - c_2 = 2 \Leftrightarrow 3 - c_2 = 2 \Rightarrow c_2 = 1$$

$$\vec{X}(t) = \frac{3}{4} \begin{pmatrix} 4 \\ 4 \end{pmatrix} e^{-3t} + \left[\begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-3t} \right].$$