Ch.6 The Laplace Transform

6.1 Definition of the Laplace Transform

Let $f(t)$ be given for $t \geq 0$ that satisfies the conditions of the theorem below. The Laplace transform of, denoted $\mathcal{L}\{f(t)\}$ or $F(s)$, is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$*$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Theorem. Suppose that

1. f is piecewise continuous for $t \geq 0$.
2. $|f(t)| \leq K e^{at}$, $t \geq M$, where K, a, M are real constants, K and M are positive.

Then, the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

Ex. let $f(t) = 1$, $t \geq 0$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\lim_{a\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^a = -\lim_{a\to\infty} \frac{e^{-sa}}{s} - \frac{1}{s}$$

$$= \frac{1}{s}, \quad s > 0.$$

Ex. $f(t) = e^{at}$, $t \geq 0$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left.\frac{e^{-(s-a)}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Ex. Let $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^1$$

$$= \frac{1 - e^{-s}}{s}, \quad s > 0$$

Ex. $f(t) = \sin(at)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \sin(at)\, dt = \frac{a}{s^2+a^2}$$

Ex. $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$, $s > 0$

Rem. $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

Ex. $\mathcal{L}\{2e^{-t} + 3\sin(2t)\} = 2\mathcal{L}\{e^{-t}\} + 3\mathcal{L}\{\sin(2t)\}$

$$= \frac{2}{s+1} + 3\,\frac{2}{s^2+4}.$$