6.2 Solution of Initial Value Problems

Thm. Suppose that f is continuous, f' is piecewise continuous. Suppose there exist constants K, a and M such that $|f(t)| \leq Ke^{at}$ for $t \geq M$.

Then, $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$ by parts

$$u = e^{-st} \qquad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \qquad v = f(t)$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sF(s)$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

$$\vdots$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

Ex. Solve the I.V.P.

$$y'' - y' - 2y = 0 \quad , y(0) = 1, \; y'(0) = 0.$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 \quad . \text{ Let } Y(s) = \mathcal{L}\{y(t)\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} - [sY(s) - \underbrace{y(0)}_{1}] - 2Y(s) = 0$$

$$(s^2 - s - 2)Y(s) - s + 1 = 0$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$Y(s) = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$$

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Ex. Solve the I.V.P. $y''+y=\sin(2t)$, $y(0)=2$, $y'(0)=1$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\sin 2t\}$$

$$s^2Y(s)-s\,y(0)-y'(0)+Y(s)=\frac{2}{s^2+4}$$

$$(s^2+1)Y(s)-2s-1=\frac{2}{s^2+4}$$

$$Y(s)=\frac{2}{(s^2+1)(s^2+4)}+\frac{2s+1}{s^2+1}$$

$$\frac{2}{(s^2+1)(s^2+4)}=\frac{as+b}{s^2+1}+\frac{cs+d}{s^2+4}$$

$$2=(as+b)(s^2+4)+(s^2+1)(cs+d)$$

$$2=as^3+4as+bs^2+4b+cs^3+ds^2+cs+d.$$

$a+c=0$, $4a+c=0$ $\Rightarrow a=c=0$

$b+d=0$ $\quad$ $4b+d=2$

$\Leftrightarrow d=-b$, $\quad 3b=2 \Rightarrow b=2/3$

$d=-2/3$

$$Y(s) = \frac{2/3}{s^2+1} - \frac{2/3}{s^2+4} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{5}{3}\frac{1}{s^2+1} + 2\frac{s}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

$$y(t) = \frac{5}{3}\sin t + 2\cos t - \frac{1}{3}\sin(2t).$$

Ex. Solve the 4th order I.V.P

$$y^{(4)} - y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0$$

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0$$

$$s^4 Y(s) - s^3\underbrace{y(0)}_{0} - s^2\underbrace{y'(0)}_{1} - s\underbrace{y''(0)}_{0} - \underbrace{y'''(0)}_{0} - Y(s) = 0$$

$$(s^4 - 1)Y(s) = s^2$$

$$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{1}{2}\frac{1}{s^2-1} + \frac{1}{2}\frac{1}{s^2+1}$$

$$y(t) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t.$$

21. $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = -1$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{4e^{-t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)Y(s) - 2s + 1 - 4 = \frac{4}{s+1}$$

$$(s+1)^2 \, Y(s) - 2s - 3 = \frac{4}{s+1}$$

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}$$

$$= \frac{4 + (2s+3)(s+1)}{(s+1)^3} = \frac{2s^2 + 5s + 7}{(s+1)^3}$$

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{c}{(s+1)^3}$$

$$2s^2 + 5s + 7 = a(s+1)^2 + b(s+1) + c.$$

$s = -1 \Rightarrow 4 = c$

diff. $4s + 5 = 2a(s+1) + b$

$s = -1 \Rightarrow 1 = b$

$4 = 2a \Rightarrow a = 2$

$$Y(s) = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$y(t) = 2e^{-t} + te^{-t} + 2t^2 e^{-t}$$

28. $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$F'(s) = \int_0^\infty e^{-st} (-t f(t))\,dt$$

$$= \mathcal{L}\{-t f(t)\}$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = -F'(s).$$

$$F''(s) = \int_0^\infty e^{-st} t^2 f(t)\,dt$$

$$= \mathcal{L}\{t^2 f(t)\}$$

$$\vdots$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

ex. $\mathcal{L}\{e^{-t}\} = \dfrac{1}{s+1}$

$$\mathcal{L}\{te^{-t}\} = -\left(\frac{1}{s+1}\right)' = \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{t^2 e^{-t}\} = \left(\frac{1}{s+1}\right)'' = \left(\frac{-1}{(s+1)^2}\right)' = \frac{2}{(s+1)^3}$$

37. $g(t) = \int_0^t f(\tau)\,d\tau$, Find $G(s)$.

$g(0) = 0$.

$g'(t) = f(t)$

$\Rightarrow \mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\}$

$sG(s) - g(0) = F(s)$

$sG(s) - 0 = F(s)$

$\Rightarrow G(s) = \dfrac{F(s)}{s}$.