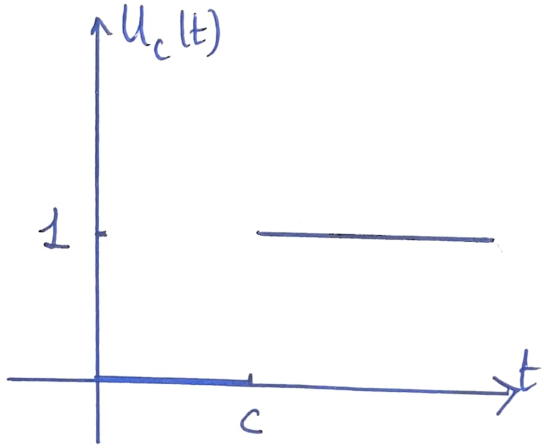
6.3 Step Functions

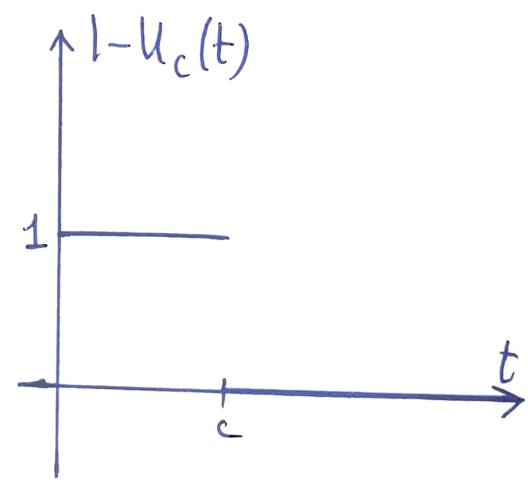
Defn. The unit step function or Heaviside function, denoted by $u_c(t)$, is defined by

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}, \quad c \geq 0.$$

$$1 - u_c(t) = 1 - \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

$$1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



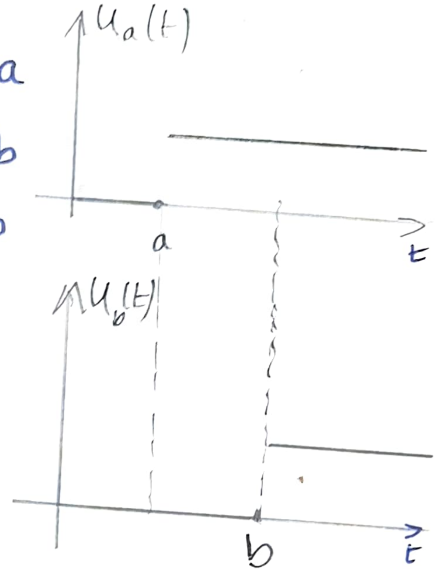


let $b > a$

$$u_a(t) - u_b(t) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases} - \begin{cases} 0 & , t < b \\ 1 & , t \geq b \end{cases}$$

$$= \begin{cases} 0-0 , & 0 \leq t < a \\ 1-0 , & a \leq t < b \\ 1-1 , & t \geq b \end{cases}$$

$$= \begin{cases} 0 & , 0 \leq t < a \\ 1 & , a \leq t < b \\ 0 & , t \geq b \end{cases}$$



Ex. Express the following functions in terms of $u_c(t)$

$$f(t) = \begin{cases} 2 & , 0 \leq t < 4 \\ 5 & , 4 \leq t < 7 \\ -1 & , 7 \leq t < 9 \\ 1 & , t \geq 9 \end{cases}$$

$$= 2(1 - u_4(t)) + 5(u_4(t) - u_7(t))$$

$$- (u_7(t) - u_9(t)) + u_9(t)$$

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

* The Laplace transform of $u_c(t)$

$$\mathcal{L}\{u_c(t)\} = \int_0^{\infty} e^{-st} u_c(t)\, dt = \int_c^{\infty} e^{-st}\, dt$$

$$= \frac{e^{-st}}{-s}\Big|_c^{\infty} = \frac{e^{-cs}}{s}, \quad s > 0.$$

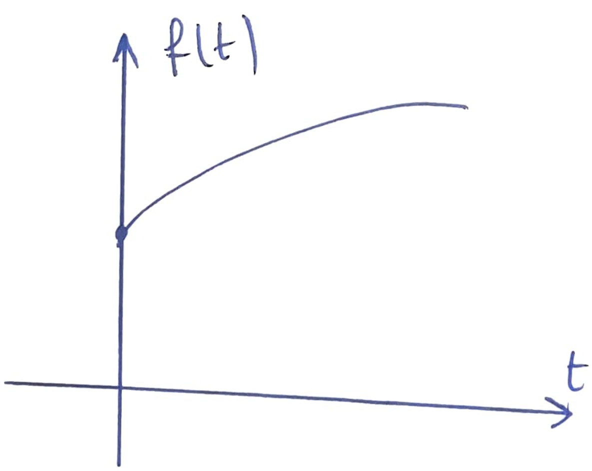
Ex. $\mathcal{L}\{u_2(t)\} = \dfrac{e^{-2s}}{s}$

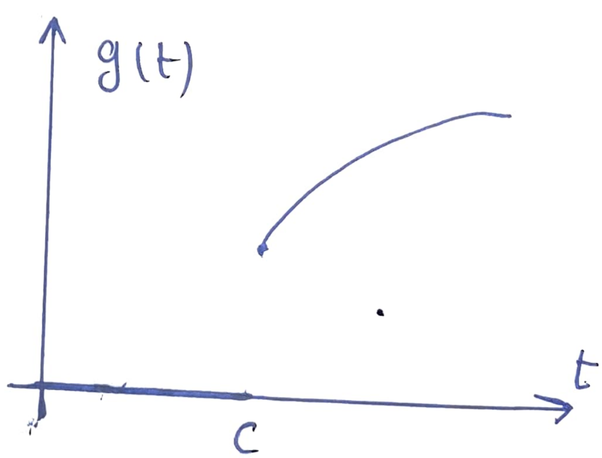
$\mathcal{L}\{u_\pi(t)\} = \dfrac{e^{-\pi s}}{s}$.

* Let $f(t)$, $t \geq 0$, be a given function. Consider the function:

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c. \end{cases}$$

$$= u_c(t)\, f(t-c)$$





$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

let $\tau = t-c$
$d\tau = dt$
$t = \tau + c$

$$= \int_0^\infty e^{-s(\tau+c)} f(\tau)\, d\tau$$

$$= e^{-cs} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= e^{-cs} F(s)$$

$$\therefore \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs} F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t-c).$$

Thm. If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \geq 0$, and c is positive const.

then, $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs} F(s)$, $s > a$.

Ex. let $f(t) = \begin{cases} \sin t & , 0 \leq t < \frac{\pi}{4} \\ \sin(t) + \cos(t - \frac{\pi}{4}) , & t \geq \frac{\pi}{4} \end{cases}$

$$= \sin t + \begin{cases} 0 & , 0 \leq t < \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & , t \geq \frac{\pi}{4} \end{cases}$$

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cos(t - \frac{\pi}{4})$$

$$F(s) = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\frac{\pi}{4}}(t) \cos(t - \frac{\pi}{4})\}$$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \mathcal{L}\{\cos t\}$$

$$F(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \frac{s}{s^2+1} = \frac{1 + se^{-\frac{\pi}{4}s}}{s^2+1}.$$

Ex. Let $F(s) = \frac{1 - e^{-2s}}{s^2} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\}$$

$$= t - u_2(t)(t-2)$$

Thm. If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \geq 0$, and if $c \geq 0$, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof. $\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt$

$$= \int_0^\infty e^{-(s-c)t} f(t)\, dt$$

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c).$$

Ex. Find the inverse transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2)$$

$$F(s) = \frac{1}{s^2+1}, \quad f(t) = \sin t$$

$$g(t) = e^{2t} f(t)$$

$$= e^{2t} \sin t.$$

14. Find the Laplace transform of

$$f(t) = \begin{cases} 0 & , \quad t < 1 \\ t^2 - 2t + 2 & , \quad t \geq 1 \end{cases}$$

$$= \begin{cases} 0 & , \quad t < 1 \\ (t-1)^2 + 1 & , \quad t \geq 1 \end{cases}$$

$$= \left((t-1)^2 + 1\right) u_1(t)$$

$$= u_1(t)\,(t-1)^2 + u_1(t)$$

$$F(s) = \mathcal{L}\{u_1(t)\,(t-1)^2\} + \mathcal{L}\{u_1(t)\}$$

$$= e^{-s}\,\mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= e^{-s}\,\frac{2}{s^3} + \frac{e^{-s}}{s} = \frac{(2+s^2)e^{-s}}{s^3}$$

20. $F(s) = \dfrac{e^{-2s}}{s^2+s-2} = \dfrac{e^{-2s}}{(s-1)(s+2)} = \left(\dfrac{1/3}{s-1} - \dfrac{1/3}{s+2}\right) e^{-2s}$

$$= \frac{1}{3}\,\frac{e^{-2s}}{s-1} - \frac{1}{3}\,\frac{e^{-2s}}{s+2} \quad ; \quad \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^{t}, \ \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$$

$$f(t) = \frac{1}{3}\left(e^{t-2} - e^{-2(t-2)}\right) u_2(t).$$

21. $F(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2} = \frac{2(s-1)e^{-2s}}{(s-1)^2+1}$

$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} = e^t \cos t$

$f(t) = 2e^{t-2}\cos(t-2)\,U_2(t)$

25-(a) $\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\,dt$ let $\tau = ct$

$d\tau = c\,dt$

$\frac{\tau}{c} = t$

$= \frac{1}{c}\int_0^\infty e^{-\left(\frac{s}{c}\right)\tau} f(\tau)\,d\tau$

$= \frac{1}{c} F\left(\frac{s}{c}\right).$

34- If $f(t+T) = f(t)$ for some $T > 0$ then f is called periodic with period T. Show that

$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

let $\tau = t - T$

$d\tau = dt$

$t = \tau + T$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(\tau+T)} \underbrace{f(\tau+T)}_{f(\tau)}\,d\tau$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\,\mathcal{L}\{f(t)\}$$

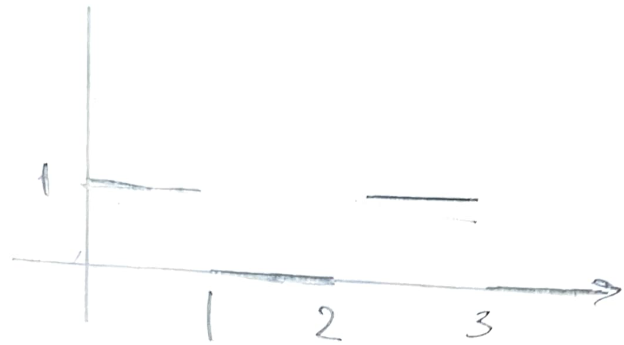
$$\left(1 - e^{-sT}\right)\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

35. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$

$f(t+2) = f(t)$

$T = 2$



$$\mathcal{L}\{f(t)\} = \frac{\int_0^2 e^{-st} f(t)\,dt}{1 - e^{-2s}}$$

$$= \frac{\int_0^1 e^{-st}\,dt}{1 - e^{-2s}} = \frac{\left.\frac{e^{-st}}{-s}\right|_0^1}{1 - e^{-2s}}$$

$$= \frac{e^{-s} - 1}{-s(1 - e^{-2s})} = \frac{1 - e^{-s}}{s(1 - e^{-s})(1 + e^{-s})}$$

$$= \frac{1}{s(1 + e^{-s})}$$