6.4 Differential Equations with Discontinuous Forcing Functions

Ex. Solve the IVP

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where $g(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5, \; t \ge 20 \end{cases}$

$$= u_5(t) - u_{20}(t)$$

$$2y'' + y' + 2y = u_5(t) - u_{20}(t), \quad y(0) = 0, \; y'(0) = 0$$

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$

$$2[s^2 Y(s) - s y(0) - y'(0)] + sY(s) - y(0) + 2Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2) Y(s) = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right)$$

$$Y(s) = \frac{1}{s(2s^2 + s + 2)}\left(e^{-5s} - e^{-20s}\right)$$

let $H(s) = \dfrac{1}{s(2s^2+s+2)} = \dfrac{a}{s} + \dfrac{bs+c}{2s^2+s+2}$

$1 = 2as^2 + as + 2a + bs^2 + cs$

$s=0 \Rightarrow 1 = 2a \Rightarrow a = \frac{1}{2}$

$0 = 2a + b \Rightarrow b = -2a = -1$

$0 = a + c \Rightarrow c = -a = -\frac{1}{2}$

$H(s) = \dfrac{1/2}{s} - \dfrac{s+\frac{1}{2}}{2(s^2+\frac{s}{2}+1)}$

$= \dfrac{1/2}{s} - \dfrac{s+\frac{1}{4}+\frac{1}{4}}{2\left[(s+\frac{1}{4})^2+\frac{15}{16}\right]}$

$= \dfrac{1/2}{s} - \dfrac{1}{2}\dfrac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2} - \dfrac{1}{8}\dfrac{4}{\sqrt{15}}\dfrac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}$

$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)$

but $Y(s) = H(s)e^{-5s} - H(s)e^{-20s}$

$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$.

Ex- Solve the IVP $y''+4y=g(t)$, $y(0)=0$, $y'(0)=0$

where $g(t)=\begin{cases} 0, & 0\le t<5 \\ \frac{t-5}{5}, & 5\le t<10 \\ 1, & t\ge 10 \end{cases}$

$$g(t)=\frac{t-5}{5}\left(u_5(t)-u_{10}(t)\right)+u_{10}(t)$$

$$=\frac{1}{5}\left[(t-5)\,u_5(t)-(t-5)u_{10}(t)+5u_{10}(t)\right]$$

$$=\frac{1}{5}\left[(t-5)u_5(t)-(t-10)\,u_{10}(t)\right]$$

$$y''+4y=\frac{1}{5}\left[(t-5)u_5(t)-(t-10)\,u_{10}(t)\right]$$

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\frac{1}{5}\left[\mathcal{L}\{(t-5)u_5(t)\}-\mathcal{L}\{(t-10)u_{10}(t)\}\right]$$

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=\frac{1}{5}\left[\frac{e^{-5s}}{s^2}-\frac{e^{-10s}}{s^2}\right]$$

$$(s^2+4)\,Y(s)=\frac{1}{5s^2}\left(e^{-5s}-e^{-10s}\right)$$

$$Y(s)=\frac{1}{5}\,\frac{1}{s^2(s^2+4)}\left(e^{-5s}-e^{-10s}\right)$$

$$\text{but } \frac{1}{s^2(s^2+4)} = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

$$Y(s) = \frac{1}{20}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)e^{-5s} - \frac{1}{20}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)e^{-10s}$$

$\frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} t$, $\quad \frac{1}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2}\sin(2t)$

$$\frac{1}{s^2+4} = \frac{2}{2(s^2+(2)^2)}$$

$$y(t) = \frac{1}{20}\left((t-5) - \frac{1}{2}\sin 2(t-5)\right)u_5(t)$$

$$- \frac{1}{20}\left((t-10) - \frac{1}{2}\sin 2(t-10)\right)u_{10}(t).$$

9. $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}$$

$$= \frac{t}{2}(1 - u_6(t)) + 3u_6(t)$$

$$= \frac{1}{2}(t - tu_6(t) + 6u_6(t))$$

$$= \frac{1}{2}(t - (t-6)u_6(t))$$

$$y'' + y = \frac{1}{2}(t - (t-6)u_6(t)), \quad y(0) = 0$$
$$y'(0) = 1$$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\frac{1}{2}\left[\mathcal{L}\{t\}-\mathcal{L}\{(t-6)u_6(t)\}\right]$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{1}{2}\left[\frac{1}{s^2}-\frac{e^{-6s}}{s^2}\right]$$

$$(s^2+1)Y(s)-1=\frac{1}{2}\left[\frac{1}{s^2}-\frac{e^{-6s}}{s^2}\right]$$

$$Y(s)=\frac{1}{2}\left[\frac{1}{s^2(s^2+1)}-\frac{1}{s^2(s^2+1)}e^{-6s}\right]+\frac{1}{s^2+1}$$

but $\displaystyle\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}$

$$Y(s)=\frac{1}{2}\frac{1}{s^2}-\frac{1}{2}\frac{1}{s^2+1}+\frac{1}{s^2+1}-\frac{1}{2}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)e^{-6s}$$

$$=\frac{1}{2}\frac{1}{s^2}+\frac{1}{s^2+1}-\frac{1}{2}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)e^{-6s}$$

$$y(t)=\frac{1}{2}\left[t+\sin t-\left[(t-6)-\sin(t-6)\right]u_6(t)\right].$$