## 6.5 Impulse Functions

In this section, we consider diff. eqs of the form

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is very large during a short time interval

$t_0 - \tau < t < t_0 + \tau$; and zero otherwise.

$g(t)$ represents a strong force or volatage acting

during very short period.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

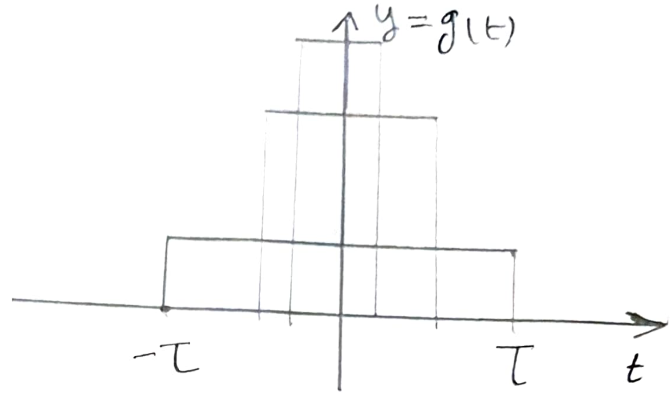
$$= \int_{-\infty}^{\infty} g(t)\,dt$$

is the total impulse of the force $g(t)$ over the

period $(t_0 - \tau,\ t_0 + \tau)$.

Suppose that $t_0 = 0$ and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & t \leq -\tau \text{, or } t \geq \tau \end{cases}$$




$$I(\tau) = \int_{-\tau}^{\tau} \frac{1}{2\tau} \, dt$$

$$= \frac{1}{2\tau} t \Big|_{-\tau}^{\tau} = \frac{1}{2\tau} \cdot 2\tau$$

$$I(\tau) = 1. \Rightarrow \lim_{\tau \to 0} I(\tau) = 1$$

and $\lim_{\tau \to 0} d_\tau(t) = 0, \quad t \neq 0$

Define an idealized unit impulse function $\delta$

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$\delta(t)$ is called Dirac delta function.

Similarly, define: $\delta(t-t_0) = 0, \quad t \neq t_0$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau \to 0} \mathcal{L}\{d_\tau(t-t_0)\}$$

$$\mathcal{L}\{d_\tau(t-t_0)\} = \int_0^{\infty} e^{-st} d_\tau(t-t_0)\,dt$$

$$= \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt$$

$$= -\frac{1}{2s\tau} \left( e^{-st} \right|_{t_0-\tau}^{t_0+\tau}$$

$$= \frac{-1}{2s\tau} \left( e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)} \right)$$

$$= e^{-st_0} \frac{e^{s\tau} - e^{-s\tau}}{2s\tau} = e^{-st_0} \frac{\sinh(s\tau)}{s\tau}$$

$$\lim_{\tau\to 0} \mathcal{L}\{d_\tau(t-t_0)\} = \lim_{\tau\to 0} \frac{\sinh(s\tau)}{s\tau}\, e^{-st_0}$$

$$= \lim_{\tau\to 0} \frac{\cancel{s}\cosh(s\tau)}{\cancel{s}}\, e^{-st_0}$$

$$= e^{-st_0}$$

$$\therefore \mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0\to 0} e^{-st_0} = 1$$

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \lim_{\tau\to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\, dt$$

$$\text{but} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\, dt = \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)\, dt$$

M.V.T. for integrals
$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ for some $a<c<b$

$$= \frac{1}{2\tau} \cdot 2\tau\, f(t^*)$$

$$= f(t^*) \,,\quad t_0-\tau < t^* < t_0+\tau$$

as $\tau \to 0$, $t^* \to t_0 \Rightarrow \int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$.

Ex. Solve the IVP. $2y''+y'+2y=\delta(t-5)$, $y(0)=0$
$y'(0)=0$

$$2\mathcal{L}\{y''\}+\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-5)\}$$

$$2\left[s^2Y(s)-sy(0)-y'(0)\right]+sY(s)-y(0)+2Y(s)=e^{-5s}$$

$$(2s^2+s+2)Y(s)=e^{-5s}$$

$$Y(s)=\frac{1}{2s^2+s+2}e^{-5s}$$

$$=\frac{1}{2\left(s^2+\frac{1}{2}s+1\right)}e^{-5s}=\frac{e^{-5s}}{2\left(\left(s+\frac{1}{4}\right)^2+\frac{15}{16}\right)}$$

$$Y(s)=\frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}e^{-5s}$$

$$y(t)=\frac{2}{\sqrt{15}}e^{-\frac{1}{4}(t-5)}\sin\frac{\sqrt{15}}{4}(t-5)\,u_5(t).$$

7 Solve the IVP $y'' + y = \delta(t-2\pi)\cos t$ , $y(0) = 0$
$y'(0) = 1$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\cos t\}$$

$$\mathcal{L}\{\delta(t-2\pi)\cos t\} = \int_0^\infty e^{-st}\,\delta(t-2\pi)\cos t\,dt$$

$$= e^{-2\pi s}\cos(2\pi) = e^{-2\pi s}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = e^{-2\pi s}$$

$$(s^2+1)Y(s) - 1 = e^{-2\pi s}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

$$y(t) = \sin t + \sin(t-2\pi)\,u_{2\pi}(t)$$

$$= \sin t + \sin t\,u_{2\pi}(t)$$

$$= \sin t\,(1 + u_{2\pi}(t)).$$