6.6 Convolution Integral

Ex. Let $H(s) = \dfrac{1}{s^2(s^2+1)}$. Find $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$$H(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$h(t) = t - \sin t.$$

We learn a new method to find $h(t)$.

Theorem. If $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)\,G(s) = \mathcal{L}\{h(t)\}, \quad s > a$$

where
$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$
$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

The function h is called the convolution of f and g, the integrals are convolution integrals.

$$h(t) = (f * g)(t)$$

Properties of Convolution

(1) $f * g = g * f$

(2) $f * (g_1 + g_2) = f * g_1 + f * g_2$

(3) $(f * g) * h = f * (g * h)$

(4) $f * 0 = 0 * f = 0.$

Rem. $1 * f \neq f.$

Ex. $1 * \cos t = \int_0^t \cos\tau \, d\tau = \sin\tau \Big|_0^t = \sin t.$

Ex. $H(s) = \frac{1}{s^2} \frac{1}{s^2+1} = F(s) G(s)$

$F(s) = \frac{1}{s^2}, \ G(s) = \frac{1}{s^2+1} \Rightarrow f(t) = t, \ g(t) = \sin t$

$$h(t) = t * \sin t = \int_0^t (t-\tau) \sin\tau \, d\tau$$

$$\begin{array}{cc} t-\tau & \sin\tau \\ -1 & -\cos\tau \\ 0 & -\sin\tau \end{array}$$

$$= -(t-\tau)\cos\tau - \sin\tau \Big|_0^t$$

$$= t - \sin t.$$

4. Find the Laplace transfor of

$$f(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$$

$$= t^2 * \cos(2t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\}\,\mathcal{L}\{\cos(2t)\}$$

$$= \frac{2}{s^3}\,\frac{s}{s^2+4}.$$

5. $$f(t) = \int_0^t (t-\tau)\, e^{\tau}\, dt$$

$$= t * e^{t}$$

$$F(s) = \mathcal{L}\{t\}\,\mathcal{L}\{e^t\} = \frac{1}{s^2}\,\frac{1}{s-1}.$$

9. $$F(s) = \frac{1}{s+1}\,\frac{s}{s^2+9} = \frac{1}{s+1}\,\frac{s}{s^2+(3)^2}$$

$$f(t) = e^{-t} * \cos(3t).$$

$$= \int_0^t e^{-(t-\tau)} \cos(3\tau)\, d\tau.$$

22. Consider the integral equation

$$\phi(t) + \int_0^t (t-\tau)\phi(\tau)\,d\tau = \sin(2t)$$

a. Solve the integral equation by Laplace transf.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\left\{\int_0^t (t-\tau)\phi(\tau)\,d\tau\right\} = \mathcal{L}\{\sin(2t)\}$$

$$\Phi(s) + \mathcal{L}\{t * \phi(t)\} = \frac{2}{s^2+4}$$

$$\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{2}{s^2+4}$$

$$\frac{s^2+1}{s^2}\Phi(s) = \frac{2}{s^2+4}$$

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)} = 2\,\frac{s}{s^2+1}\,\frac{s}{s^2+(2)^2}$$

$$\phi(t) = 2\cos t * \cos 2t$$

$$= 2\int_0^t \cos(t-\tau)\cos(2\tau)\,d\tau$$

(b) Convert the integral equation to diff. eq.

$$\phi(t) + \int_0^t (t-\tau)\phi(\tau)\,d\tau = \sin 2t$$

$$\phi(0) = 0.$$

$$\phi'(t) + \int_0^t \phi(\tau)\,d\tau = 2\cos(2t)$$

$$\phi'(0) = 2$$

$$\phi''(t) + \phi(t) = -4\sin(2t), \quad \phi(0)=0, \quad \phi'(0)=2$$

$$s^2\Phi(s) - s\phi(0) - \phi'(0) + \Phi(s) = -4\frac{2}{s^2+4}$$

$$(s^2+1)\Phi(s) - 2 = \frac{-8}{s^2+4}$$

$$(s^2+1)\Phi(s) = \frac{-8}{s^2+4} + 2$$

$$= \frac{2s^2}{s^2+4}$$

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)},$$

26. $\phi'(t) + \int_0^t (t-\tau)\phi(\tau)\,d\tau = t,\ \phi(0) = 0.$

a. $s\Phi(s) - \phi(0) + \mathcal{L}\{t\}\mathcal{L}\{\Phi(t)\} = \mathcal{L}\{t\}$

$$s\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{1}{s^2}$$

$$\frac{s^3+1}{s^2}\Phi(s) = \frac{1}{s^2}$$

$$\Phi(s) = \frac{1}{s^3+1} = \frac{1}{s+1}\,\frac{1}{s^2-s+1}$$

$$= \frac{1/3}{s+1} + \frac{as+b}{s^2-s+1}$$

$$\frac{1}{s^3+1} = \frac{1}{3}(s^2-s+1) + (as+b)(s+1)$$

$$1 = \left(\frac{1}{3}+a\right)s^2 + \left(-\frac{1}{3}+a+b\right)s + \frac{1}{3} + b$$

$$a + \frac{1}{3} = 0 \Rightarrow a = -\frac{1}{3}$$

$$-\frac{1}{3} + a + b = 0 \Rightarrow b = \frac{1}{3} - a = \frac{2}{3}$$

$$\Phi(s) = \frac{1/3}{s+1} + \frac{-\frac{1}{3}s + \frac{2}{3}}{s^2-s+1} = \frac{1}{3}\left[\frac{1}{s+1} - \frac{s-2}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

$$= \frac{1}{3}\left[\frac{1}{s+1} - \frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{2}\,\frac{2}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$\phi(t) = \frac{1}{3}\left[e^{-t} - e^{t/2}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\,e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).\right.$$

b. Convert the integro-diff. eq. to diff. eq.

$$\phi'(t) + \int_0^t (t-\tau)\phi(\tau)\,d\tau = t, \quad \phi(0)=0, \quad \phi'(0)=0$$

diff. $$\phi''(t) + \int_0^t \phi(\tau)\,d\tau = 1$$

$$\phi''(0) = 1$$

diff. $$\phi'''(t) + \phi(t) = 0$$

$$s^3\Phi(s) - s^2\phi(0) - s\phi'(0) - \phi''(0) + \Phi(s) = 0$$

$$(s^3+1)\Phi(s) - 1 = 0$$

$$\Phi(s) = \frac{1}{s^3+1}$$