Ch.1 Introduction

Many principles or laws underlying the behaviour of the natural world are relations involving rates. The relations are called equations and the rates are derivatives.

Defn. Differential equations are equations that contain derivatives.

A differential equation that describes some physical process is called a mathematical model of the process.

Ex. Motion of fluids, flow of current in an electric circuit, dissipation of heat in solid objects, the propagation and detection of seismic waves, the decrease or increase of a population.

Ex. 1. Falling object in the atmosphere near sea level.

t: time , v: velocity m/s

m: mass kg , a: acceleration m/s^2

Two forces act on the object,

drag force $= \gamma v$, $kg \cdot m/s^2$

γ constant kg/s ,

gravity force mg , $g = 9.8\ m/s^2$

Write a differential equation for $v(t)$.

We assume that v is positive in the downward direction

The physical law that governs the motion of objects is Newton's second law.

let F be the net force on the object.

$F = ma$

but $F = mg - \gamma v$, $a = \frac{dv}{dt}$

$$mg - \gamma v = m\frac{dv}{dt}$$

∴ a mathematical model of an object falling in the atmosphere near sea level is:

$$m\frac{dv}{dt} = mg - \gamma v$$

A function $v(t)$ that satisfies this equation is called a solution of the equation.

v is called the dependent variable

t is called the independent variable

m, $g = 9.8\ m/s^2$, γ are constants.

Differential equations are classified according to:

(1) order of the highest derivative in the equation.

(2) Linearity with respect to the dependent variable and its derivatives.

The equation $m \frac{dv(t)}{dt} = mg - \gamma v$ is a first order linear differential equation.

Assume that $v(0) = v_0$ is the initial velocity

Then, $v(0) = v_0$ is called an initial condition (I.C.) and

$$m \frac{dv}{dt} = mg - \gamma v \quad , \quad v(0) = v_0$$

is called an initial value problem.

(I.V.P.)

We can study the properties of the solutions before finding them, using direction field and phase diagram.

Let $m = 10\ kg$, $\gamma = 2\ kg/s$, $g = 9.8\ m/s^2$

$$m \frac{dv}{dt} = mg - \gamma v \Rightarrow 10 \frac{dv}{dt} = (10)(9.8) - 2v$$

$$\frac{dv}{dt} = 9.8 - \frac{1}{5} v.$$

- Equilibrium solution is found by setting

$$\frac{dv}{dt} = 0 \Rightarrow 9.8 - \frac{1}{5} v = 0$$

$$v(t) = 49\ m/s$$

it is a solution of the diff. equation.

at the equilibrium, the body falls at a constant speed $49\ m/s$ when there is a balance between the two forces acting on the object

$$mg = \gamma v.$$

$\frac{dv}{dt} = 9.8 - \frac{1}{5}v$. Notice that, for each value of v, we can find $\frac{dv}{dt}$ using the above equation.

If $v = 49$ m/s $\Rightarrow \frac{dv}{dt} = 0$

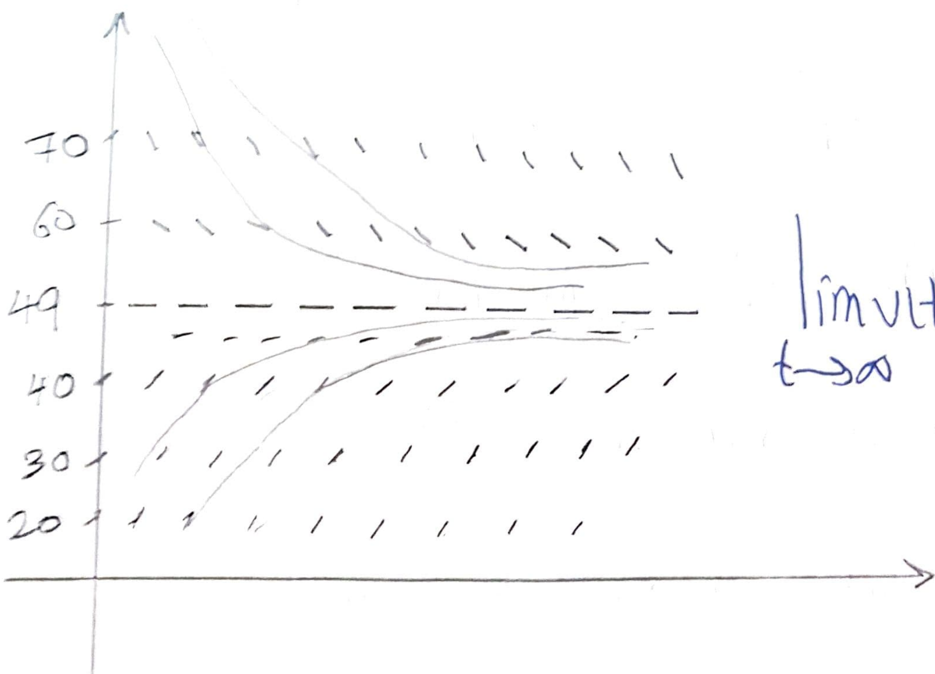
$v = 40 \Rightarrow \frac{dv}{dt} = \frac{1}{5}(49-40) = \frac{9}{5} = 1.8$

$v = 30 \Rightarrow \frac{dv}{dt} = \frac{1}{5}(49-30) = \frac{19}{5} = 3.8$

$v = 50 \Rightarrow \frac{dv}{dt} = \frac{1}{5}(49-50) = \frac{-1}{5} = -0.2$

$v = 60 \Rightarrow \frac{dv}{dt} = \frac{1}{5}(49-60) = \frac{-11}{5} = -2.2$

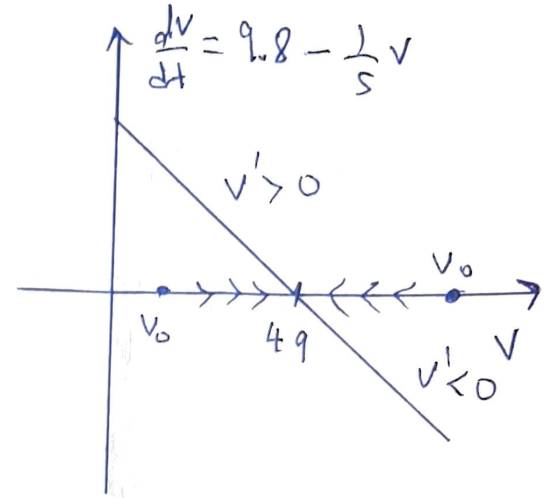
$\frac{dv}{dt}$ is the slope of the solution.



$\lim_{t\to\infty} v(t) = 49$ m/s

We can use also the phase diagram to show that $\lim_{t \to \infty} v(t) = 49$. We plot the equation

$$\frac{dv}{dt} = 9.8 - \frac{1}{5}v$$



Finally, we find the solution $v(t)$.

$$\frac{dv}{dt} = \frac{1}{5}(49 - v) \Rightarrow \frac{dv}{49 - v} = \frac{1}{5}dt$$

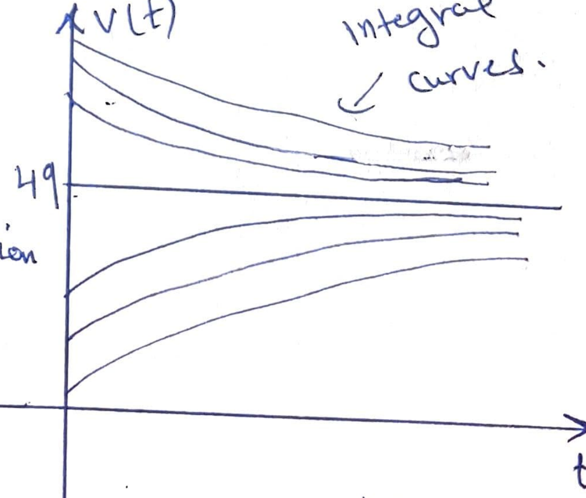
$$\int \frac{dv}{v - 49} = -\frac{1}{5}\int dt \Rightarrow \ln|v(t) - 49| = -\frac{1}{5}t + c$$

$$|v(t) - 49| = e^{-\frac{1}{5}t + c} = e^{c}e^{-\frac{1}{5}t}$$

$$v(t) - 49 = \pm e^{c}e^{-\frac{t}{5}}$$

$$v(t) = 49 + Ce^{-t/5}$$

general solution



$$\lim_{t \to \infty} v(t) = 49$$

$$v(0) = v_0 = 49 + C \Rightarrow C = v_0 - 49$$

Solution of the IVP $v(t) = 49 + (v_0 - 49)e^{-t/5}$

Ex. 2. Field Mice and Owls

let $p(t)$ be the mouse population which increases at a rate proportional to $p(t)$ with proportionality factor r called the rate Constant or growth rate.

$$\frac{dp}{dt} = r p(t),$$

where t is time in month

Assume that several owls live in the same area that kill 15 mice per day, i.e, 450 mice/month

let $r = \frac{1}{2}$, then,

$$\frac{dp}{dt} = \frac{1}{2} p(t) - 450 \quad , \quad p(0) = p_0$$

Equilibrium: $\frac{dp}{dt} = 0 = \frac{1}{2} p(t) - 450$

$$\Rightarrow p(t) = 900.$$

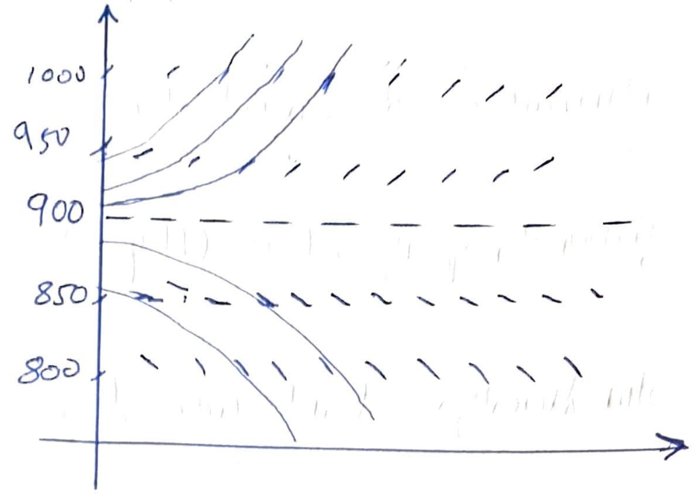
Direction Field: $\frac{dP}{dt} = \frac{1}{2}p(t) - 450$

$P = 800 \Rightarrow \frac{dP}{dt} = -50$

$P = 850 \Rightarrow \frac{dP}{dt} = -25$

$P = 950 \Rightarrow \frac{dP}{dt} = 25$

$P = 1000 \Rightarrow \frac{dP}{dt} = 50$

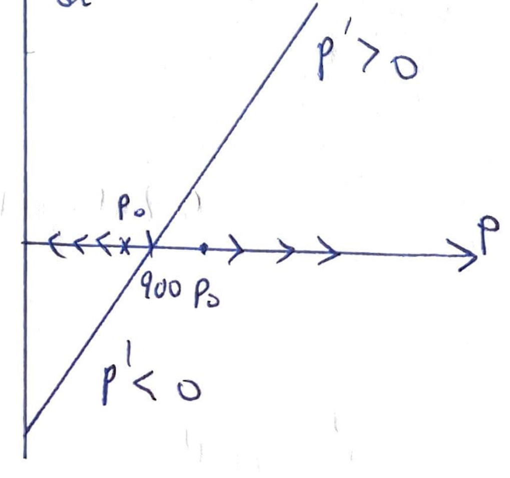


If $P_0 > 900 \Rightarrow \lim_{t \to \infty} p(t) = +\infty$

If $P_0 < 900 \Rightarrow \lim_{t \to \infty} p(t) = -\infty$

Since $p(t) \geq 0$, if $P_0 < 900 \Rightarrow$ there exists T such that $p(T) = 0$. "extinction".

* Phase Diagram:



Solution of the IVP $\frac{dp}{dt} = \frac{1}{2}p - 450, \; p(0) = p_0$

$$\frac{dp}{dt} = \frac{1}{2}(p-900) \Rightarrow \frac{dp}{p-900} = \frac{1}{2}dt$$

$$\int \frac{dp}{p-900} = \frac{1}{2}\int dt$$

$$\ln|p(t)-900| = \frac{1}{2}t + c$$

$$|p(t)-900| = e^{\frac{1}{2}t+c} = e^{c}e^{t/2}$$

$$p(t) - 900 = \pm e^{c}e^{t/2} = Ce^{t/2}$$

$$p(t) = 900 + Ce^{t/2}$$

$$p(0) = p_0 = 900 + C \Rightarrow C = p_0 - 900$$

$$p(t) = 900 + (p_0 - 900)e^{t/2}$$

Assume $p_0 = 800$. Find T such that $P(T) = 0$

$$0 = 900 - 100e^{T/2}$$

$$e^{T/2} = 9 \Rightarrow \frac{T}{2} = \ln 9 \Rightarrow T = 2\ln 9$$

$$T = \ln 81.$$

We have derived the diff. eqs

$$m\frac{dv}{dt} = mg - \gamma v$$

$$\frac{dp}{dt} = r p(t) - k$$

These equations are of the form $\frac{dy}{dt} = ay - b$
$= a\left(y - \frac{b}{a}\right)$

equilibrium $y = \frac{b}{a}$

$$\int \frac{dy}{y - \frac{b}{a}} = a \int dt$$

$$\ln\left|y - \frac{b}{a}\right| = at + c$$

$$y(t) = \frac{b}{a} + c e^{at}.$$

if $a < 0 \Rightarrow \lim_{t\to\infty} y(t) = \frac{b}{a}$

if $y(0) = y_0 \Rightarrow y_0 = \frac{b}{a} + c$

$$\Rightarrow c = y_0 - \frac{b}{a}$$

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}.$$

Classification of differential equations

$\frac{dy}{dt} = 2y+1$ 1st order linear.

$y' = y^2$ 1st order nonlinear.

$z'' + z = e^t$ 2nd order linear.

$w\frac{d^2w}{dx^2} + x = 1$ 2nd order nonlinear.

$y''' + y'' + y' + y = 0$ 3rd order linear.

$y'' = y + (y')^2$ 2nd order nonlinear.

ex. Show that $y = \sin t$ is a solution of $y'' + y = 0$

$y' = \cos t, \quad y'' = -\sin t$

$y'' + y = -\sin t + \sin t = 0$.