10.7 Power Series

Defn. A power series about $x=0$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

A power series about $x=a$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Ex. $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

in this series $c_0 = 1$, $c_1 = 1$, $\ldots$, $c_n = 1$ $\forall n$

this is a geometric series with ratio $r = x$

$\Rightarrow$ the series converges if $|x| < 1$

and $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$, $r = x$, $|x| < 1$

We can use the partial sums, which are polynomials to approximate the function $\frac{1}{1-x}$.

$P_0(x) = 1$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + x^2$

$\vdots$

$P_n(x) = 1 + x + x^2 + \cdots + x^n$.

Ex. $\sum_{n=0}^{\infty} \left(-\frac{(x-2)}{2}\right)^n = 1 - \frac{x-2}{2} + \frac{(x-2)^2}{4} - \cdots$

this is a power series centered at $x = a = 2$

which is a geometric series with ratio $r = -\frac{(x-2)}{2}$

in this series, $c_0 = 1$, $c_2 = -\frac{1}{2}$, $c_3 = \frac{1}{4}$, $\ldots$, $c_n = \left(-\frac{1}{2}\right)^n$

the series converges if $\left|-\frac{(x-2)}{2}\right| < 1$

$\Leftrightarrow \quad \left|\frac{x-2}{2}\right| < 1 \Leftrightarrow -1 < \frac{x-2}{2} < 1 \Leftrightarrow -2 < x-2 < 2$

$\Leftrightarrow 0 < x < 4$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x}, \quad 0 < x < 4$$

Ex. For what values of x do the following power series Converge!

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Apply the ratio test to $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{x^n}{n} \right| = \sum_{n=1} \frac{|x|^n}{n}$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \frac{n}{|x|^n} = |x| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x|$$

$\Rightarrow$ the series Converges absolutely for $|x| < 1$ and diverges if $|x| > 1$ (since $u_n \not\to 0$)

if $|x| = 1 \Rightarrow x = \pm 1$

if $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ alternating harmonic series which Converges by AST.

if $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$

which diverges since $\sum \frac{1}{n}$ is the harmonic series $(p=1)$

$\therefore$ the series Converges if $-1 < x \leq 1$

- Converges absolutely if $-1 < x < 1$

Converges Conditionally at $x = 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2(n+1)-1}}{2(n+1)-1} \cdot \frac{2n-1}{x^{2n-1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right|$$

$$= x^2 \lim_{n \to \infty} \frac{2n-1}{2n+1}$$

$$= x^2$$

$\Rightarrow$ the series Converges absolutely if $x^2 < 1$

and diverges if $x^2 > 1$.

if $x^2 = 1 \Rightarrow x = \pm 1$

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \quad \text{Converges by AST.}$$

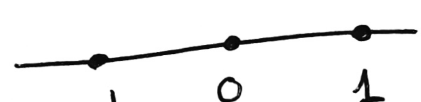
$$x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{3n-2}}{2n-1}$$

$$= \sum \frac{(-1)^{3n} (-1)^{-2}}{2n-1} = \sum_{n=1}^{\infty} \frac{\left((-1)^3\right)^n}{2n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \quad \text{Converges by AST.}$$

$\Rightarrow$ the series Converges if $-1 \leq x \leq 1$

" Converges absolutely if $-1 < x < 1$



Converges Conditionally if $x = \pm 1$

Since $\sum \left| \frac{(-1)^n}{2n-1} \right| = \sum \frac{1}{2n-1}$ diverges by L.C.T with $\sum \frac{1}{n}$.

(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{|x|\cancel{|x|^n}}{(n+1)\cancel{n!}} \frac{\cancel{n!}}{\cancel{|x|^n}}$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |x| \cdot 0 = 0 < 1, \text{ for all } x$$

$\therefore$ The series Converges absolutely for all x

(d) $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + \cdots$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,\cancel{|x|^n}\,|x|}{\cancel{n!}\,\cancel{|x|^n}}$$

$$= |x| \lim_{n\to\infty} (n+1) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{if } x \neq 0 \end{cases}$$

the series Converges at $x=0$ only.

Radius and Interval of Convergence:

The Convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is described by one of the following cases:

1. There is a positive number R such that the series Converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$. The series may or may not Converge for $|x-a|=R$ $\Leftrightarrow$ $x=a-R$, $x=a+R$ "endpoints"

2. The series Converges absolutely for every x
   $R=\infty$

3. The series Converges at $x=a$ and diverges for $x \neq a$, $R=0$.

R: is called the radius of Convergence and the interval Centered at $x=a$ is Called the interval of Convergence.

To find the interval and radius of Convergence, we use the ratio or root test for $\sum |u_n|$

- If the interval is finite, we check Convergence at the endpoints.

For the above examples:

	Series	Interval of Conv.	Radius R
(a)	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 < X \leq 1$	$R = 1$
(b)	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 \leq X \leq 1$	$R = 1$
(c)	$\sum_{n=1}^{\infty} \frac{x^n}{n!}$	$-\infty < X < \infty$	$R = \infty$
(d)	$\sum_{n=0}^{\infty} n!\, x^n$	Conv. at $X = 0$	$R = 0$

* Operations on Power Series

1- Series Multiplication

Thm. If $\sum_{n=0}^{\infty} a_n x^n = A(x)$, $\sum_{n=0}^{\infty} b_n x^n = B(x)$

Converges absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{\infty} a_k b_{n-k}$$

$\Rightarrow \sum_{n=0}^{\infty} c_n x^n$ Converges absolutely to $A(x)B(x)$

for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

and $\sum_{n=0}^{\infty} c_n x^n = A(x)\,B(x)$ for $|x| < R$.

Ex. Find $\left(\sum_{n=0}^{\infty} x^n\right)^2$.

$$\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \left(1+x+x^2+\cdots\right)\left(1+x+x^2+\cdots\right)$$

$$\Rightarrow \quad a_0 = a_1 = a_2 = \cdots = 1$$

$$b_0 = b_1 = b_2 = \cdots = 1$$

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

$$= \underbrace{1 + 1 + \cdots + 1}_{(n+1)\text{-times}}$$

$$= n+1$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} x^n\right) = \sum c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

but $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) x^n = \left(\sum_{n=0} x^n\right)^2 = \frac{1}{(1-x)^2}$$

for $|x|<1$

Theorem. If $\sum_{n=0}^{\infty} a_n x^n$ Converges absolutely for $|x|<R \Rightarrow \sum_{n=0}^{\infty} a_n (f(x))^n$ Converges absolutely for any Continuous function $f(x)$ on $|f(x)|<R$.

Ex. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}$ for $|-x|<1$

$\sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}$, $|2x|<1$

$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1-x^2}$, $x^2<1$

Term by term differentiation:

Thm. If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad a-R<x<a+R$$

Then, $f(x)$ has derivatives of all orders, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) (x-a)^{n-2}$$

$$\vdots$$

and each series at every point in $a-R<x<a+R$.

Ex. $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} \quad , \quad |x|<1$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) x^n$$

Term by term integration

Thm. Suppose that $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ Converges

for $a-R < x < a+R$, $R>0$. Then

$$\int f(x)\,dx = \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1} + C$$

Converges for $a-R < x < a+R$.

Ex. Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots$$ geometric series $r = -x^2$

$$= \frac{1}{1+x^2}$$

$$\Rightarrow f'(x) = \frac{1}{1+x^2},$$

$$f(x) = \int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1}x + C$$

but $f(x) = x - \frac{x^3}{3} + \cdots$

$$f(0) = 0 = \tan^{-1}0 + C \Rightarrow C = 0$$

$$\Rightarrow f(x) = \tan^{-1}x.$$

Ex. $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots, \quad -1 < t < 1$

$$\int_0^x \frac{dt}{1+t} = \int_0^x [1 - t + t^2 - t^3 + \cdots]\,dt$$

$$\ln(1+t)\Big|_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right|_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \quad -1 < x < 1$$

10.8 Taylor and Maclaurin Series

Defn. Let f be a function with derivatives of all orders in some interval containing a as interior point. The Taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin Series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Ex. Find the Taylor Series generated by $f(x) = \frac{1}{x}$ at $a=1$

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2!\,x^{-3}$, $\cdots$ $f^{(n)}(x) = (-1)^n n!\,x^{-n-1}$

$f(1) = 1$, $f'(1) = -1$, $f''(1) = 2!$, $f'''(1) = -3!$, $\cdots$ $f^{(n)}(1) = (-1)^n n!$

$\therefore f^{(k)}(1) = (-1)^k k! \Rightarrow \frac{f^{(k)}(1)}{k!} = (-1)^k$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} (-1)^k (x-1)^k = 1 - (x-1) + (x-1)^2 - \cdots$$

$$1-(x-1)+(x-1)^2 - \cdots + (-1)^n (x-1)^n + \cdots = \frac{1}{1+(x-1)} = \frac{1}{x}$$

if $|x-1|<1$.

$a=2$. Find Taylor Series of $f(x)=\frac{1}{x}$

$$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$$

$$f^{(n)}(2) = (-1)^n n!\, 2^{-(n+1)} = \frac{(-1)^n n!}{2^{n+1}}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}}(x-2)^k$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-3)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

$$= \frac{1}{1+\frac{x-2}{2}} = \frac{1}{x}, \quad \left|\frac{x-2}{2}\right| < 1$$

$$|x-2| < 2$$

Taylor polynomial of order n

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$: Linearization

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Remark. We speak of Taylor polynomial of order n and not of degree n because $f^{(n)}(a)$ may be zero.

Ex. Find the Maclaurin Series of $f(x) = e^x$.

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$P_0(x) = 1$, $P_1(x) = 1 + x$, $P_2(x) = 1 + x + \frac{x^2}{2!}$

Ex. Find the Maclaurin series of $f(x) = \cos x$

$f(x) = \cos x$, $f'(x) = -\sin x$

$f''(x) = -\cos x$ $f'''(x) = \sin x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \cos x$, $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

$f^{(2n)}(0) = (-1)^n$, $f^{(2n+1)}(0) = 0.$

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} - \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$P_0(x) = P_1(x) = 1$

$P_2(x) = P_3(x) = 1 - \frac{x^2}{2!}$

$\vdots$

$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$

2. Find the Maclaursin Series of $f(x) = \sin x$.

$f(x) = \sin x \qquad f'(x) = \cos x$

$f''(x) = -\sin x \qquad f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x \qquad f^{(5)}(x) = \cos x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \sin x \,, \qquad f^{(2n+1)}(x) = (-1)^n \cos x$

$f^{(2n)}(0) = 0 \qquad f^{(2n+1)}(0) = (-1)^n$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$P_1(x) = P_2(x) = x$

$P_3(x) = P_4(x) = x - \frac{x^3}{3!}$

19. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots$

$= 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$

$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

30. $f(x) = 2^x$, $a = 0$, $a = 1$

$2^x = e^{\ln 2^x} = e^{(\ln 2)x} = \sum_{n=0}^{\infty} \frac{((\ln 2)x)^n}{n!}$

$= 1 + (\ln 2)x + \frac{(\ln 2)^2}{2!} x^2 + \frac{(\ln 2)^3}{3!} x^3 + \cdots$

$2^x = 2 \cdot 2^{x-1} = 2e^{\ln 2^{x-1}} = 2e^{(x-1)\ln 2}$

$= 2\sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} (x-1)^k$

Using definition.

$$f(x) = 2^x, \quad a = 1$$

$$f(1) = 2, \quad f'(x) = 2^x \ln 2, \quad f''(x) = 2^x (\ln 2)^2$$

$$f^{(n)}(x) = (\ln 2)^n 2^x$$

$$f^{(n)}(1) = 2(\ln 2)^n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = 2\sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!}(x-1)^k$$

5.1 Review of Power Series

A power series centered at $x = x_0$ is a series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

The series converges if $\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n$ exists for that x. It converges for $x = x_0$.

The series converges absolutely if

$$\sum_{n=0}^{\infty} |a_n (x - x_0)^n| \text{ converges}$$

Apply the ratio test

$$\lim_{n \to \infty} \left| \frac{a_{n+1} (x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= |x - x_0| L$$

The series converges absolutely if $|x-x_0|L<1$

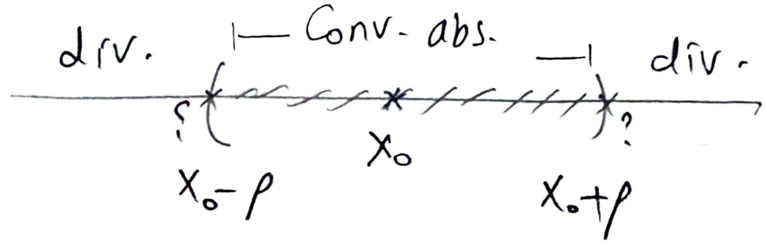
and diverges if $|x-x_0|L>1$

it may or may not converge if $|x-x_0|L=1$.

* There is a nonnegative number ρ, called the radius of convergence such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely if $|x-x_0|<\rho$. This interval is called the interval of convergence of the series.

the series diverges if $|x-x_0|>\rho$,

it may converge or diverge if $|x-x_0|=\rho$



Ex. Find the radius and interval of Convergence of

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 2^n}.$$

Apply the ratio test

$$\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}\,\frac{n 2^n}{(x+1)^n}\right| = \frac{|x+1|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x+1|}{2}$$

$\Rightarrow$ the series Conv. abs. if $\frac{|x+1|}{2} < 1$

$\Rightarrow |x+1| < 2 \Rightarrow \rho = 2$

diverges if $|x+1| > 2$

if $|x+1| = 2 \Rightarrow x+1 = 2$ or $x+1 = -2$

if $x+1 = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges : harmonic Series

$x = 1$

$x+1 = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Converges by alternating Series test.

$x = -3$

Interval $-3 \leq x < 1$

If $f(x)$ is infinitely differentiable, the Taylor series generated by f at x_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

f is called analytic if it has convergent power series in some interval.

Rem. Power series can be differentiated, integrated term by term.

we can also, add, subtract, multiply and divide two power series.

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

$$\Leftrightarrow a_n = b_n, \quad n = 0, 1, 2, 3, \ldots$$