# Solutions - Sec. 1.1

Write a differential equation of the form $\frac{dy}{dt} = ay + b$ such that, as $t \to \infty$

7. All solutions approach $y = 2$

equilibrium $y = -\frac{b}{a} = 2 \Rightarrow b = -2a$, and $a < 0$

for example, $a = -2$, $b = 4$

$$\frac{dy}{dt} = -2y + 4.$$

or $a = -1$, $b = 2$, $\frac{dy}{dt} = -y + 2$.

10. All other solutions diverge from $y = \frac{1}{2}$

$-\frac{b}{a} = \frac{1}{2} \Rightarrow a = -2b$ and $a > 0$

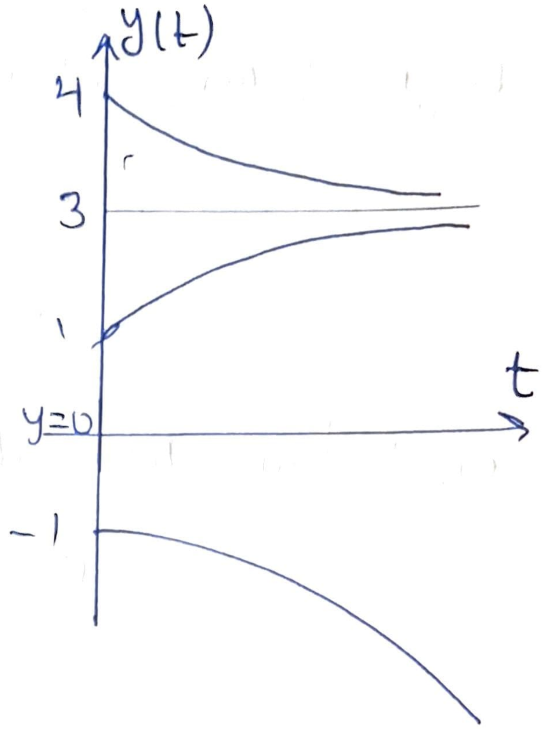
for example, $a = 1$, $b = -\frac{1}{2}$

$$\frac{dy}{dt} = y - \frac{1}{2}$$

or $a = 2$, $b = -1$

$$\frac{dy}{dt} = 2y - 1.$$

11. $y' = y(3-y)$.




equilibria: $y' = 0 = y(3-y)$

$\Rightarrow y = 0,\ y = 3$

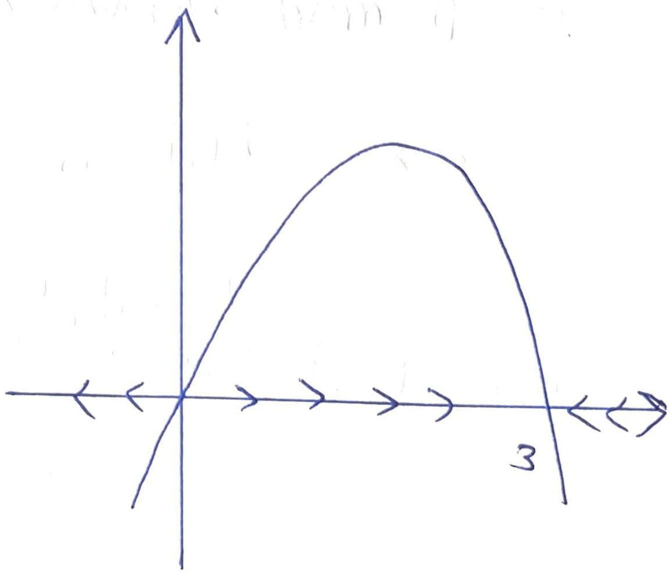
if $y = -1 \Rightarrow y' = -4$

$\Rightarrow \lim_{t\to\infty} y(t) = -\infty$

If $y = 1$, $y' = 2 > 0 \Rightarrow \lim_{t\to\infty} y(t) = 3$

If $y = 4$, $y' = -4 < 0 \Rightarrow \lim_{t\to\infty} y(t) = 3$

Phase diagram




$y=3$ is stable

$y=0$ is unstable.

22. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume as a function of time.

$$\frac{dV}{dt} \propto S \quad \text{where } S = 4\pi r^2$$

$$\frac{dV}{dt} = k(4\pi r^2)$$

$$\text{but } V = \frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3}{4\pi}V\right)^{1/3}$$

$$\frac{dV}{dt} = 4k\pi \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}$$

$$\frac{dV}{dt} = AV^{2/3}$$ for some constant $A$.

this is a first order nonlinear diff eq.

23. Newton's law of Cooling: The temperature of an object changes at a rate proportional to the difference between the temperature of the object and the temperature of its surrounding

let $u(t)$ : temp. of the object

$T$ : surrounding temperature,

$$\frac{du}{dt} = -k(u - T).$$

let $k = 0.05 \text{ min}^{-1}$, $T = 20°C$

$$\frac{du}{dt} = -0.05(u - 20), \quad 1^{st} \text{ order linear.}$$

$$\int \frac{du}{u-20} = -0.05 \int dt$$

$$\ln|u-20| = -0.05t + c$$

$$u(t) = 20 + Ce^{-0.05t} \rightarrow 20 \text{ as } t \to \infty.$$

24. (Drug example): Fluid Containing 5 mg / cm$^3$ of drug flows into a patient's blood at a rate of 100 cm$^3$/h. drug is absorbed or leaves the bloodstream at a rate proportional to the amount present with rate constant 0.4 h$^{-1}$. Let $q(t)$ be the quantity of drug present at time t.

a. Write a diff. eq. for $q(t)$.

$$\frac{dq}{dt} = 5 \frac{mg}{cm^3} * 100 \frac{cm^3}{h} - 0.4 \frac{1}{h} q(t)(mg).$$

$$\frac{dq}{dt} = 500 - 0.4\, q(t)$$ : 1$^{st}$ order linear.

b. How much of the drug present in the blood after a long time.

eq. $\frac{dq}{dt} = 0 = 500 - 0.4 q.$

$$q = \frac{500}{0.4} = 1250 \text{ mg}.$$

Solutions of sect.2


11. falling object

$$m\frac{dv}{dt} = mg - \gamma v^2 ,$$

$m = 10$ kg, $g = 9.8\ m/s^2$, limiting velocity $v_L = 49m$

$$0 = \frac{dv}{dt} = (10)(9.8) - \gamma (49)^2$$

$$\Rightarrow \gamma = \frac{98}{(49)^2}$$

$$10\frac{dv}{dt} = 98 - \frac{98}{(49)^2}v^2$$

$$10\frac{dv}{dt} = \frac{98}{(49)^2}\left((49)^2 - v^2\right) = \frac{2}{49}\left((49)^2 - v^2\right)$$

$$\frac{dv}{dt} = \frac{1}{245}\left((49)^2 - v^2\right)$$

this is a first order nonlinear diff eq.

$$\int \frac{dv}{(49)^2 - v^2} = \int \frac{dt}{245}$$

Using partial fractions to solve $\int \frac{dv}{(49)^2 - v^2}$.

$$\frac{1}{(49)^2 - v^2} = \frac{1}{(49-v)(49+v)}$$

$$= \frac{1/98}{49-v} + \frac{1/98}{49+v}$$

$$\frac{1}{98} \int \frac{dv}{49-v} + \int \frac{dv}{49+v} = \frac{1}{245} \int dt$$

$$-\ln|49-v| + \ln|49+v| = \frac{98}{245}(t+c)$$

$$\ln\left|\frac{49+v}{49-v}\right| = \frac{2}{5}t + c_1$$

$$\left|\frac{49+v}{49-v}\right| = e^{c_1} e^{\frac{2}{5}t}$$

$$\frac{49+v}{49-v} = Ce^{\frac{2}{5}t}$$

$$49+v = Ce^{\frac{2}{5}t}(49-v)$$

$$v(1+Ce^{\frac{2}{5}t}) = 49(Ce^{2/5t} - 1) \Rightarrow v(t) = \frac{49(Ce^{\frac{2}{5}t} - 1)}{Ce^{\frac{2}{5}t} + 1}.$$

15. $\frac{du}{dt} = -k(u-T)$ , $u(0) = u_0$

a. $\int \frac{du}{u-T} = -k\int dt$

$\ln|u-T| = -kt + c$

$|u-T| = e^{-kt+c} = e^{c}e^{-kt}$

$u - T = \pm e^{c} e^{-kt} \Rightarrow u(t) = T + Ce^{-kt}$

$u(0) = u_0 = T + C \Rightarrow C = u_0 - T$

$u(t) = T + (u_0 - T)e^{-kt}$

$\therefore u(t) - T = (u_0 - T)e^{-kt}$.

b. $\tau$ time at which $u(t) - T = \frac{1}{2}(u_0 - T)$

$\frac{1}{2}(u_0 - T) = (u_0 - T)e^{-k\tau}$

$\frac{1}{2} = e^{-k\tau}$

$\ln\frac{1}{2} = -k\tau$

$\ln 2 = k\tau$