Ch. 3 Second Order Linear Equations

3.1 Homogeneous Equations with Constant Coefficients

The general form of 2nd order diff. eqs is

$$\frac{d^2y}{dt^2} = f(t, y, y')$$

The general form of 2nd order linear eqs

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

if $g(t) = 0$, then, the equation

$$y'' + p(t)\,y' + q(t)\,y = 0$$

is called homogeneous.

in sec. 3.1, 3.3, 3.4, we find solutions of

second order eqs with constant coeffs:

$$ay'' + by' + cy = 0.$$

IVPs of 2nd order has the IC

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

$y = e^{rt}$ is a solution of $ay'' + by' + cy = 0$

$\Leftrightarrow a(r^2 e^{rt}) + b(re^{rt}) + ce^{rt} = 0$

$\Leftrightarrow (ar^2 + br + c)e^{rt} = 0$

$\Leftrightarrow ar^2 + br + c = 0$

This equation is called the characteristic equation. This is a quadratic equation that has 2 roots: $r_{1,2} = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

(1) if $b^2 > 4ac$, the roots are real, distinct.

(2) if $b^2 < 4ac$, the roots are complex.

(3) if $b^2 - 4ac = 0$, the roots are repeated

$r_1 = r_2 = -\dfrac{b}{2a}$

If $b^2-4ac>0$, then r_1, r_2 are real, $r_1 \neq r_2$

$\Rightarrow y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ are two solutions.

the general solution of the eq. is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Ex. $y'' - y = 0$, $y(0) = 2$, $y'(0) = -1$

$$r^2 - 1 = 0 \Rightarrow r = \pm 1$$

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

$$y'(t) = c_1 e^{t} - c_2 e^{-t}$$

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = c_1 - c_2 = -1$$

$$\Rightarrow c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}$$

$$y(t) = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$$

Ex. $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = +3$

$r^2 + 5r + 6 = 0 \Leftrightarrow (r+2)(r+3) = 0$

$r_1 = -2$, $r_2 = -3$

$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$y(0) = c_1 + c_2 = 2$ —(1)

$y'(0) = -2c_1 - 3c_2 = 3$ —(2)

$3(1) + (2) \Rightarrow c_1 = 9$, $(1) \Rightarrow c_2 = -7$

$y(t) = 9e^{-2t} - 7e^{-3t}$

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (9e^{-2t} - 7e^{-3t}) = 0$

Ex. $4y'' - 8y' + 3y = 0$

$4r^2 - 8r + 3 = 0 \Leftrightarrow (2r-3)(2r-1) = 0$

$r_1 = 3/2$, $r_2 = 1/2$

$y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{t/2}$

12. $y''+3y'=0$, $y(0)=-2$, $y'(0)=3$

$r^2+3r=0 \Rightarrow r(r+3)=0 \Rightarrow r=0,\ r=-3$

$y(t) = c_1 + c_2 e^{-3t}$

$y'(t) = -3c_2 e^{-3t}$

$y(0) = c_1 + c_2 = -2$

$y'(0) = -3c_2 = 3 \Rightarrow c_2 = -1$

$\Rightarrow c_1 = -1$

$y(t) = -1 - e^{-3t}$

17. Find a diff eq. whose solution is $y = c_1 e^{2t} + c_2 e^{-3t}$

$r_1 = 2,\ r_2 = -3$

$(r_1 - 2)(r_2 + 3) = 0 \Rightarrow r^2 + r - 6 = 0$

$y'' + y' - 6y = 0$.

21. Solve the IVP $y''-y'-2y=0$, $y(0)=\alpha$, $y'(0)=2$.

Find α so that $y(t)\to 0$ as $t\to\infty$

charact. eq. $r^2-r-2=0 \Leftrightarrow (r-2)(r+1)=0$

$r_1=2$, $r_2=-1$

$y(t)=c_1e^{2t}+c_2e^{-t}$

$y(t)\to 0$ as $t\to\infty$ if $c_1=0$

$y'(t)=2c_1e^{2t}-c_2e^{-t}$

$y(0)=c_1+c_2=\alpha$

$y'(0)=2c_1-c_2=2$

Adding up. $\Rightarrow 3c_1=\alpha+2$

$c_1=\dfrac{\alpha+2}{3}$

$\therefore$ $c_1=0 \Leftrightarrow \alpha=-2$.