3.2 Solutions of Linear Homogeneous Equations

In this section, we study the structure of solutions of 2nd order linear homog. eqs.

$$y'' + p(t)y' + q(t)\,y = 0$$

Thm. (Existence and Uniqueness Thm)

Consider the IVP

$$y'' + p(t)\,y' + q(t)y = g(t) \quad y(t_0) = y_0, \; y'(t_0) = y_0'$$

If p, q and g are continuous in an open interval I that contains the point t_0, then, there is exactly one solution $y = \phi(t)$ of this problem and the solution is defined throughout I.

Ex. Find the longest interval in which the solution of the IVP :

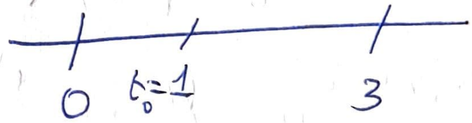
$$(t^2-3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

$$y' + \frac{t}{t(t-3)}y' - \frac{t+3}{t(t-3)}y = 0$$

$$p(t) = \frac{1}{t-3}, \quad q(t) = -\frac{t+3}{t(t-3)} \quad \text{Conts, if } t \neq 0, 3$$

$$I = (0, 3).$$

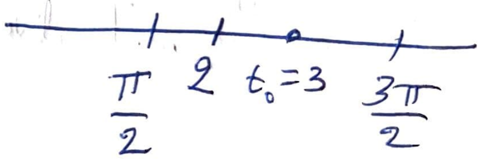


12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

$$y'' + \frac{1}{x-2}y' + \frac{\sin x}{\cos x}y = 0$$

$$p(x) = \frac{1}{x-2}, \quad x \neq 2, \quad q(x) = \frac{\sin x}{\cos x}, \quad x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$I = \left(2, \frac{3\pi}{2}\right).$$



If y_1 and y_2 are solutions of $y''+p(t)y'+q(t)y=0$

$y_1''+p(t)y_1'+q(t)y_1=0$ — (1)

$y_2''+p(t)y_2'+q(t)y_2=0$ — (2)

$\Leftrightarrow$ $c_1y_1''+p(t)c_1y_1'+q(t)c_1y=0$ — (3)

$c_2y_2''+p(t)c_2y_2'+q(t)c_2y_2=0$ — (4)

$\therefore$ c_1y_1 and c_2y_2 are also solutions

(3) + (4) $\Rightarrow$

$$(c_1y_1+c_2y_2)''+p(t)(c_1y_1+c_2y_2)'+q(t)(c_1y_1+c_2y_2)=0$$

$\Rightarrow$ $c_1y_1+c_2y_2$ is a solution.

Thm. (Principle of Superposition)

If y_1 and y_2 are solutions of the diff eq.

$$y''+p(t)y'+q(t)y=0$$

$\Rightarrow$ the linear combination $c_1y_1+c_2y_2$ is also a solution for any constants c_1 and c_2.

Let $y(t) = c_1 y_1(t) + c_2 y_2(t)$, Can we find c_1 and c_2

such that $y(t_0) = y_0$, $y'(t_0) = y_0'$

$$y'(t) = c_1 y_1'(t) + c_2 y_2'(t)$$

c_1 and c_2 are solution of the system

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

This system has a unique solution iff

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

$$= y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)$$

$$\neq 0$$

$W(y_1, y_2)(t_0)$ is called the Wronskian of

y_1 and y_2 at t_0.

Thm. Suppose that y_1 and y_2 are solutions of the diff eq. $y'' + p(t)y' + q(t)y = 0$

and that the initial Conditions

$$y(t_0) = y_0 \ , \quad y'(t_0) = y_0'$$

are assigned. Then, it is always possible to choose Constants C_1, C_2 such that

$$y = C_1 y_1(t) + C_2 y_2(t)$$

Satisfies the diff eq. and the I.Cs

if and only if $W(y_1, y_2)(t_0) \neq 0$

Ex. $y'' + 5y' + 6y = 0$

charct. eq. $r^2 + 5r + 6 = 0 \iff (r+2)(r+3) = 0$

$r_1 = -2$, $r_2 = -3$, $y_1 = e^{-2t}$, $y_2(t) = e^{-3t}$

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -3e^{-5t} + 2e^{-5t} = -e^{-5t} \neq 0$$

Thm. Suppose that y_1 and y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = 0$$

Then, the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of the diff. eq. if and only if there is a point t_0 such that $W(y_1, y_2)(t_0) \neq 0$.

Rem. If $W(y_1, y_2)(t_0) \neq 0$ for some t_0 then

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary constants c_1 and c_2 is called the general solution of the diff eq. and y_1, y_2 are said to form a fundamental set of solutions.

Ex.

$$W(e^{r_1 t}, e^{r_2 t}) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t}$$

Ex. $y_1(t) = t^{1/2}$, $y_2(t) = t^{-1}$ are solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0$$

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2} t^{-3/2} \neq 0$$

$\therefore$ $t^{1/2}, t^{-1}$ is a Fundamental set of solutions.

Thm. Consider the diff. eq. $y'' + p(t) y' + q(t) y = 0$

where p, q are conts in some interval I.

Let $t_0 \in I$. Let y_1 be the solution of the diff. eq.

the satisfies the IC

$$y(t_0) = 1, \quad y'(t_0) = 0$$

and y_2 be a solution that satisfies the IC

$$y(t_0) = 0, \quad y'(t_0) = 1$$

$\Rightarrow$ y_1, y_2 form a F.S. of solutions.

Proof $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Ex. Find the F.S. of solutions specified the the theorem for the diff eq: $y'' - y = 0$

$$r^2 - 1 = 0 \Leftrightarrow (r-1)(r+1) = 0 \Rightarrow r = \pm 1$$

$$y(t) = c_1 e^{t} + c_2 e^{-t}, \quad y'(t) = c_1 e^{t} - c_2 e^{-t}$$

$$y(0) = 1, \; y'(0) = 0 \Rightarrow \begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{array} \Rightarrow c_1 = c_2 = \frac{1}{2}$$

$$y_1(t) = \frac{e^{t} + e^{-t}}{2} = \cosh t$$

$$y(0) = 0, \; y'(0) = 1$$

$$\begin{array}{l} c_1 + c_2 = 0 \\ c_1 - c_2 = 1 \end{array} \Rightarrow c_1 = \frac{1}{2}, \; c_2 = -\frac{1}{2}$$

$$y_2(t) = \frac{e^{t} - e^{-t}}{2} = \sinh t$$

F.S. of Solutions $\{\cosh t, \sinh t\}$

Theorem (Abel's Theorem)

If y_1 and y_2 are solutions of the diff. eq.

$$y'' + p(t)\,y' + q(t)\,y = 0$$

where p and q are continuous on an open interval I, then, the Wronskian is given by

$$W(y_1, y_2)(t) = c\,e^{-\int p(t)\,dt}$$

where c is a constant that depends on y_1 and y_2 but not on t.

$W(y_1, y_2)(t) = 0$ for all t if $c = 0$

$W(y_1, y_2)(t) \neq 0$ for all t if $c \neq 0$.

Ex. $2t^2 y'' + 3ty' - y = 0$, $t > 0$

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$$

$$W = c\,e^{-\int \frac{3}{2t}dt} = c e^{-\frac{3}{2}\ln t} = c t^{-3/2}$$

in a previous example, $y_1 = t^{1/2}$, $y_2 = t^{-1}$

$W(y_1, y_2) = -\frac{3}{2} t^{-3/2} \Rightarrow c = -\frac{3}{2}$ for the FS of solns

Proof. Let y_1, y_2 be solutions of $y'' + p(t)y' + q(t)y = 0$

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad - (1)$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad - (2)$$

$-y_2 * (1) + y_1 * (2) \Rightarrow$

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0$$

$$W' + p(t) W = 0$$

$$\frac{dW}{W} = -p(t)$$

$$\Rightarrow \ln|W| = -\int p(t)\,dt + c$$

$$W(t) = c e^{-\int p(t)\,dt}$$

#.

Rem. $W = y_1 y_2' - y_1' y_2$

$$W' = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_1' y_2'} - y_1'' y_2$$

$$= y_1 y_2'' - y_1'' y_2.$$

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18. If $W(f,g) = t^2 e^t$, $f(t) = t$. Find $g(t)$.

$$W(f,g) = fg' - f'g = t^2 e^t$$

$$tg' - g = t^2 e^t$$

$$\Rightarrow g' - \frac{1}{t} g = te^t \text{, linear equation}$$

$$\mu(t) = e^{-\int \frac{dt}{t}} = e^{-\ln t} = \frac{1}{t}$$

$$g(t) = t\left[\int e^t dt + c\right] = te^t + ct.$$

34. y_1, y_2 - F.S. of solutions of $ty'' + 2y' + te^t y = 0$

$W(y_1, y_2)(1) = 3$. Find $W(y_1, y_2)(5)$.

$$y'' + \frac{2}{t} y' + e^t y = 0 \Rightarrow p(t) = \frac{2}{t}$$

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt} = ce^{-\int \frac{2}{t} dt} = ce^{-2\ln t}$$

$$= \frac{c}{t^2}$$

$$W(y_1, y_2)(1) = 3 = c \Rightarrow W(y_1, y_2)(t) = \frac{3}{t^2}$$

$$W(y_1, y_2)(5) = \frac{3}{25}.$$

3.3 Complex Roots of the charact. eq.

We Consider the eq. $ay''+by'+cy=0$

whose charact. eq. is $ar^2+br+c=0$

if $b^2-4ac<0$, the equation has Complex roots

$r_1=\lambda+i\mu$, $r_2=\lambda-i\mu$

$e^{(\lambda+i\mu)t}=e^{\lambda t}e^{i\mu t}$, $e^{i\theta}=\cos\theta+i\sin\theta$

$=e^{\lambda t}\cos(\mu t)+i\,e^{\lambda t}\sin(\mu t)$

we have two real solutions of the diff eq.

$y_1(t)=e^{\lambda t}\cos(\mu t)$, $y_2(t)=e^{\lambda t}\sin(\mu t)$

The general solution is

$$y(t)=c_1e^{\lambda t}\cos(\mu t)+c_2e^{\lambda t}\sin(\mu t).$$

Ex. $y''+y=0$

$r^2+1=0 \Rightarrow r=\pm i \Rightarrow \lambda=0,\ \mu=1$

$$y(t)=C_1\cos t+C_2\sin t$$

Ex. $y'' + y' + 9.25y = 0$

$r^2 + r + 9.25 = 0$

$r^2 + r + \frac{1}{4} + 9 = 0$

$(r + \frac{1}{2})^2 = -9 \Rightarrow r + \frac{1}{2} = \pm 3i$

$r = -\frac{1}{2} \pm 3i$, $\lambda = -\frac{1}{2}$, $\mu = 3$

$y(t) = c_1 e^{-t/2} \cos(3t) + c_2 e^{-t/2} \sin(3t)$

7. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0 \Leftrightarrow r - 2r + 1 + 1 = 0$

$(r-1)^2 + 1 = 0 \Rightarrow r - 1 = \pm 1 \Rightarrow r = 1 \pm i$, $\lambda = 1$, $\mu = 1$

$y(t) = c_1 e^t \cos t + c_2 e^t \sin t$.

22. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0 \Leftrightarrow (r^2 + 2r + 1) + 1 = 0$

$(r+1)^2 + 1 = 0 \Rightarrow (r+1)^2 = -1 \Rightarrow r + 1 = \pm i$

$r = -1 \pm i$, $\lambda = -1$, $\mu = 1$

$y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$.

3.4 Repeated Roots, Reduction of Order

To solve the diff. eq. $ay'' + by' + cy = 0$

we find the roots of the characteristic equation

$$ar^2 + br + c = 0$$

if $b^2 - 4ac = 0$ then, this equation has a

repeated root $r = r_1 = r_2 = -\frac{b}{2a}$

$y_1(t) = e^{-\frac{b}{2a}t}$ is a solution.

A second solution is given by:

$$y_2(t) = t e^{-\frac{b}{2a}t}$$

Ex. $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0 \iff (r+2)^2 = 0 \Rightarrow r = -2$

$y_1(t) = e^{-2t}$, $y_2(t) = te^{-2t}$

The general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Ex. Solve the IVP.

$$y'' - y' + \frac{1}{4}y = 0 \quad , \quad y(0) = 2, \; y'(0) = \frac{1}{3} .$$

$$r^2 - r + \frac{1}{4} = 0$$

$$\left(r - \frac{1}{2}\right)^2 = 0 \Rightarrow r = \frac{1}{2}$$

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}$$

$$y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{t}{2}} + \frac{1}{2} c_2 t e^{t/2}$$

$$y(0) = c_1 + c_2(0) = 2 \Rightarrow c_1 = 2$$

$$y'(0) = \frac{1}{2} c_1 + c_2 = \frac{1}{3}$$

$$c_1 = 2 \Rightarrow 1 + c_2 = \frac{1}{3} \Rightarrow c_2 = -\frac{2}{3}$$

$$y(t) = 2 e^{\frac{t}{2}} - \frac{2}{3} t e^{t/2}$$

Reduction of Order

Suppose that $y_1(t)$ is a solution of the diff eq.

$$y'' + p(t)y' + q(t)y = 0 \quad — \quad (1)$$

$$\Rightarrow y_1'' + p(t)y_1' + q(t)y_1 = 0$$

A second solution has the form $y = v(t)y_1(t)$

for some function $v(t)$.

This method is called the reduction of order

$$y(t) = v(t)\,y_1(t)$$
$$y' = vy_1' + v'y_1$$
$$y'' = vy_1'' + 2v'y_1' + y_1v''$$

Substitute in (1) and rearrange the terms:

$$y_1v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{0}v = 0$$

$$\Rightarrow y_1v'' + (2y_1' + py_1)\,v' = 0$$

this is a second order eq. in v

We solve this eq. to find $v(t)$:

$$y_1 v'' + (2y_1' + p y_1) v' = 0$$

$$\frac{v''}{v'} = -\left(\frac{2y_1'}{y_1} + p\right)$$

$$\int \frac{dv'}{v'} = -2\int \frac{dy_1}{y_1} - \int p(t)\,dt$$

$$\ln v' = -2\ln y_1 - \int p(t)\,dt$$

$$= \ln y_1^{-2} - \int p(t)\,dt$$

$$v'(t) = \frac{1}{y_1^2} e^{-\int p(t)\,dt}$$

$$v(t) = \int \frac{1}{y_1^2} e^{-\int p(t)\,dt}\,dt$$

$\therefore$ The second solution is

$$y_2(t) = v(t)\,y_1(t).$$

Ex. Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0$$

Find a second solution $y_2(t)$.

$$y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0$$

$$p(t) = \frac{3}{2t}$$

$$v(t) = \int \frac{1}{y_1^2} e^{-\int p(t)\, dt} dt$$

$$= \int t^2 e^{-\int \frac{3}{2t} dt} dt = \int t^2 e^{-\frac{3}{2}\ln t} dt$$

$$= \int t^2 \frac{1}{t^{3/2}} dt = \int t^{1/2} dt = \frac{2}{3} t^{3/2}$$

$$y_2(t) = v(t)\, y_1(t) = \frac{2}{3} t^{3/2} \cdot \frac{1}{t} = \frac{2}{3} t^{1/2}$$

we can take $y_2(t) = t^{1/2}$

The general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{1/2}.$$

12. $y'' - 6y' + 9y = 0, \; y(0) = 0, \; y'(0) = 2$

$r^2 - 6r + 9 = 0 \Leftrightarrow (r-3)^2 = 0 \Rightarrow r = 3$

$y(t) = c_1 e^{3t} + c_2 t e^{3t}$

$y(0) = c_1 = 0$

$y'(t) = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$

$y'(0) = 3c_1 + c_2 = 2,$

$c_1 = 0 \Rightarrow c_2 = 2$

$\Rightarrow y(t) = 2te^{3t}.$

26. $t^2 y'' - t(t+2) y' + (t+2) y = 0, \; t > 0, \; y_1(t) = t$

Find $y_2(t)$.

$y'' - \frac{t+2}{t} y' + \frac{t+2}{t^2} y = 0$

$p(t) = -\left(1 + \frac{2}{t}\right), \quad e^{-\int p(t)dt} = e^{\int (1+\frac{2}{t})dt} = e^t e^{2\ln t}$

$= t^2 e^t$

$v(t) = \int \frac{1}{y_1^2} e^{-\int p(t)dt} dt = \int \frac{1}{t^2} t^2 e^t dt$

$= \int e^t dt = e^t$

$y_2(t) = te^t$

$y(t) = c_1 t + c_2 t e^t.$

Sec. 3.4

33. If $y_1 \neq 0$ is a solution of $y'' + p(t)y' + q(t)y = 0$

Show that y_2 satisfies $\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2}$

$$y_1 y_2' - y_1' y_2 = W(y_1, y_2)$$

$$\Rightarrow \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$$

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2}$$

$$\left(\frac{y_2}{y_1}\right)' = \frac{e^{-\int p(t)\,dt}}{y_1^2}$$

$$\frac{y_2}{y_1} = \int \frac{1}{y_1^2} e^{-\int p(t)\,dt}\, dt$$

$$y_2(t) = y_1 \int \frac{1}{y_1^2} e^{-\int p(t)\,dt}\, dt.$$

34. $(x-1)y'' - xy' + y = 0, \; x>1, \; y_1(x) = e^x$

Find $y_2(x)$.

$$y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$$

$$W(y_1, y_2)(x) = e^{-\int p(x)\,dx} = e^{\int \frac{x}{x-1} dx}$$

$$= e^{\int \left(1 + \frac{1}{x-1}\right) dx} = e^{x + \ln(x-1)}$$

$$= e^{\ln(x-1)} e^x = (x-1)\,e^x$$

$$y_2(x) = e^x \int \frac{1}{e^{2x}} (x-1) e^x \, dx$$

$$= e^x \int (x-1) e^{-x} \, dx$$

$x-1$	$\xrightarrow{+}$	e^{-x}
1	$\xrightarrow{-}$	$-e^{-x}$
0		e^{-x}

$$= \cancel{e^x}\left[-(x-1)\cancel{e^{-x}} - \cancel{e^{-x}}\right]$$

$$= -x + 1 - 1$$

$$= -x$$

$$y_2(x) = x.$$