3.5 Nonhomogeneous Equations

Method of Undetermined Coefficients

We consider the nonhomog. equation

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{—— (1)}$$

Suppose that Y_1 and Y_2 are two solutions of this equation.

Let y_1, y_2 be a fundamental set of solution of the corresponding homog. eq.

$$y'' + p(t)y' + q(t)y = 0 \quad \text{—— (2)}$$

Since Y_1, Y_2 are two solutions of (1) then

$$Y_1'' + p(t)Y_1' + q(t)Y_1 = g(t) \quad \text{—— (3)}$$

$$Y_2'' + p(t)Y_2'' + q(t)Y_2 = g(t+1 \quad \text{—— (4)}$$

$$(3)-(4) \Rightarrow (Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0$$

$\therefore$ $Y_1 - Y_2$ is a solution of homog. eq. (2)

since y_1, y_2 is a F.S. of solutions of (2)

$\therefore \quad Y_1 - Y_2 = c_1 y_1 + c_2 y_2$

$Y_1 = Y_2 + c_1 y_1 + c_2 y_2$

Thus, we get the form of solutions of the nonhomog. eq. (1).

If Y_p is a particular solution of (1) and y_1, y_2 is a F.S. of solutions of (2) then, the general solution of the nonhomog. eq. (1) is

$$Y(t) = Y_p(t) + c_1 y_1(t) + c_2 y_2(t)$$

$$= Y_p(t) + y_c(t)$$

Y_p : is called a particular solution

$y_c(t) = c_1 y_1 + c_2 y_2$ is called Complementary Solution.

Method of Undetermined Coefficients:

It is a method for finding $Y_p(t)$. This method can be used if $g(t)$ is exponential, sine, cosine, polynomial or combination of these functions.

Ex Find the general solution of $y'' - 3y' + 2y = 3e^{-t}$

homog. eq. $y'' - 3y' + 2y = 0$

charact. eq. $r^2 - 3r + 2 = 0 \Leftrightarrow (r-1)(r-2) = 0$

$\Rightarrow r_1 = 1,\ r_2 = 2,\ y_1(t) = e^t,\ y_2(t) = e^{2t}$

$y_c(t) = c_1 e^t + c_2 e^{2t}$

$Y_p(t) = Ae^{-t},\ Y_p'(t) = -Ae^{-t},\ Y_p''(t) = Ae^{-t}$

subst. into the eq.

$Ae^{-t} - 3(-Ae^{-t}) + 2Ae^{-t} = 3e^{-t}$

$6Ae^{-t} = 3e^{-t} \Rightarrow 6A = 3 \Rightarrow A = \frac{1}{2}$

$Y_p(t) = \frac{1}{2}e^{-t}$ is a particular solution.

general solution: $y(t) = \frac{1}{2}e^{-t} + c_1 e^t + c_2 e^{2t}$.

Ex. Find a particular solution of the diff. eq:

$$y'' - 3y' + 2y = 2\sin t$$

$$Y_p(t) = A\sin t + B\cos t$$

$$Y_p' = A\cos t - B\sin t$$

$$Y_p'' = -A\sin t - B\cos t$$

Subst. $-A\sin t - B\cos t - 3(A\cos t - B\sin t)$

$$+ 2(A\sin t + B\cos t) = 2\sin t$$

$$(A + 3B)\sin t + (-3A + B)\cos t = 2\sin t$$

(1) $A + 3B = 2$

(2) $-3A + B = 0 \Rightarrow B = 3A.$

(1) $\Rightarrow A + 9A = 2 \Rightarrow 10A = 2 \Rightarrow A = \frac{1}{5}$

$\Rightarrow B = \frac{3}{5}$

$$Y_p(t) = \frac{1}{5}\sin t + \frac{3}{5}\cos t$$

$$y(t) = \frac{1}{5}\sin t + \frac{3}{5}\cos t + c_1 e^{t} + c_2 e^{2t}.$$

Ex. Find a particular solution of $y'' - 3y' + 2y = 2t^2 + 3$

$$Y_p(t) = At^2 + Bt + C$$

$$Y_p'(t) = 2At + B, \quad Y_p''(t) = 2A$$

Subst. $2A - 3(2At + B) + 2(At^2 + Bt + C) = 2t^2 + 3$

$$2At^2 + (-6A + 2B)t + 2A - 3B + 2C = 2t^2 + 3$$

Coeff. of t^2: $2A = 2 \Rightarrow A = 1$

Coeff. of t: $-6A + 2B = 0 \Rightarrow B = 3A = 3$

Constant: $2A - 3B + 2C = 3$

but $A = 1, B = 3 \Rightarrow 2 - 9 + 2C = 3 \Rightarrow 2C = 10 \Rightarrow C = 5$

$$Y_p(t) = t^2 + 3t + 5.$$

general solution: $y(t) = t^2 + 3t + 5 + c_1 e^t + c_2 e^{2t}$

Ex. Find $Y_p(t)$ for $y'' - 3y' + 2y = e^t \cos t$.

general form is:

$$Y_p(t) = Ae^t \cos t + Be^t \sin t.$$

Find Y_p', Y_p'' and substitute to find A and B.

Ex. Find the general form of $Y_p(t)$ for

$$y'' - 3y' + 2y = t \cos t.$$

$$Y_p(t) = (At+B)\sin t + (Ct+D)\cos t.$$

Find Y_p', Y_p'' and subst. to find A, B, C, D.

Ex. Find the general form of $Y_p(t)$ for

$$y'' - 3y' + 2y = t^2 e^{4t}$$

$$Y_p(t) = (At^2 + Bt + C)e^{4t}.$$

Ex. Find the general form of $Y_p(t)$ for

$$y'' - 3y' + 2y = \cos^2 t$$

$$\Leftrightarrow y'' - 3y' + 2y = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

$$Y_p(t) = A + B\cos(2t) + C\sin(2t).$$

Ex. Find $Y_p(t)$ for $y''-3y'+2y=2e^t$

if $Y_p(t)=Ae^t \Rightarrow Y_p'=Y_p''=Ae^t$

Subst. $Ae^t-3e^t+2e^t=2e^t$

$0=2e^t$ X

the form we took above is not Correct because

e^t is a solution of the homog. eq.

The Correct form of $Y_p(t)$ is

$Y_p(t)=Ate^t$

$Y_p'(t)=Ae^t+Ate^t$

$Y_p''(t)=Ae^t+Ae^t+Ate^t$

$=2Ae^t+Ate^t$

Subst. $2Ae^t+\cancel{Ate^t}-3(Ae^t+\cancel{Ate^t})+2\cancel{Ate^t}=2e^t$

$-Ae^t=2e^t \Rightarrow -A=2 \Rightarrow A=-2$

$Y_p(t)=-2te^t$

$y(t)=-2te^t+c_1e^t+c_2e^{2t}$.

Ex. Find the form of $Y_p(t)$ of $y''+y=\sin t$.

homog. eq. $y''+y=0$

Charact. eq. $r^2+1=0 \Rightarrow r=\pm i$

$y_c(t)=c_1\cos t+c_2\sin t$

$Y_p(t)=At\sin t+Bt\cos t$.

6. $y''+2y'+y=2e^{-t}$

$y''+2y'+y=0$

Charact. eq. $r^2+2r+1=0 \Leftrightarrow (r+1)^2=0$.

$r=-1 \Rightarrow y_c(t)=c_1e^{-t}+c_2te^{-t}$

$Y_p(t)=At^2e^{-t}$.

12. $y''-y'-2y=3\cosh(2t)$

$y''-y'-2y=\frac{3}{2}e^{2t}+\frac{3}{2}e^{-2t}$

$y''-y'-2y=0 \Rightarrow r^2-r-2=0 \Leftrightarrow (r-2)(r+1)=0$

$r=2,\ r=-1$

$y_c(t)=c_1e^{2t}+c_2e^{-t}$

$Y_p(t)=Ate^{2t}+Be^{-2t}$

18. $y'' + 2y' + 5y = 4e^{-t}\cos(2t)$

$y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0 \Leftrightarrow (r+1)^2 + 4 = 0 \Leftrightarrow r = -1 \mp 2i$

$y_c(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

$Y_p(t) = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$.

21. $y'' + y = t + t\sin t$

$y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c(t) = c_1 \cos t + c_2 \sin t$

$Y_p(t) = At + B + t(Ct + D)\sin t + t(Et + F)\cos t$.

Ex. $y'' + y' = 1 + e^{-t}$

$y'' + y' = 0 \Rightarrow r^2 + r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r_1 = 0,\ r_2 = -1$

$y_c(t) = c_1 + c_2 e^{-t}$

$Y_p(t) = At + Bte^{-t}$.