3.6 Variation of Parameters

Suppose that $y_1(t)$ and $y_2(t)$ are a Fundamental set of Solutions of the homog. equation

$$y'' + p(t)y' + q(t)y = 0$$

According to the Method of Variation of Parameters a particular solution of the nonhomog. eq.

$$y'' + p(t)y' + q(t)y = g(t)$$

has the form

$$Y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

where, $$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

$$u_2(t) = \int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

Where $W(y_1,y_2)(t) = y_1y_2' - y_2y_1'$ is the Wronskian.

Ex. Find a particular solution of $y''+4y = 2\csc t$

$$y''+4y=0 \Rightarrow r^2+4=0 \Rightarrow r=\pm 2i$$

$$y_1(t) = \cos(2t) \quad , \quad y_2(t) = \sin(2t)$$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

$$= 2\left(\cos^2(2t) + \sin^2(2t)\right) = 2$$

$$u_1(t) = -\int \frac{\sin(2t)\, 2\csc t}{2}\, dt = -\int \frac{\sin(2t)}{\sin t}\, dt$$

$$= -\int \frac{2\sin t \cos t}{\sin t}\, dt = -2\int \cos t\, dt = -2\sin t$$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\cos 2t = -2\sin^2 t + 1$

$$u_2(t) = \int \frac{\cos(2t)\, 2\csc t}{2}\, dt = \int \frac{\cos(2t)}{\sin t}\, dt$$

$$= \int \frac{1-2\sin^2 t}{\sin t}\, dt = \int \csc t\, dt - 2\int \sin t\, dt$$

$$= -\ln|\csc t + \cot t| + 2\cos t.$$

$$Y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$= -2\sin t \cos(2t) + \left(-\ln|\csc t + \cot t| + 2\cos t\right)\sin(2t)$$

5. $y'' + y = \tan t$

$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_1(t) = \cos t \,,\; y_2(t) = \sin t$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \sin t \tan t \, dt$$

$$= -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{1 - \cos^2 t}{\cos t} dt$$

$$= -\int \sec t \, dt + \int \cos t \, dt$$

$$= -\ln|\sec t + \tan t| + \sin t.$$

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)} dt = \int \cos t \tan t \, dt$$

$$= \int \cos t \frac{\sin t}{\cos t} dt = \int \sin t \, dt = -\cos t$$

$$Y_p = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

$$= [-\ln|\sec t + \tan t| + \sin t] \cos t - \cos t \sin t$$

$$= -\ln|\sec t + \tan t| \cos t$$

17. $x^2y'' - 3xy' + 4y = x^2\ln x$, $x > 0$, $y_1(x) = x^2$, $y_2(x) = x^2\ln x$

$\Leftrightarrow\ y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x$

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix}$$

$$= x^3 + 2x^3\ln x - 2x^3\ln x$$

$$= x^3$$

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W(x)}dx = -\int \frac{x^2\ln x \cdot \ln x}{x^3}dx$$

$$= -\int \frac{(\ln x)^2}{x}dx \quad \text{by subst. } u = \ln x$$

$$= -\frac{1}{3}(\ln x)^3$$

$$u_2(x) = \int \frac{y_1(x)g(x)}{W(x)}dx = \int \frac{x^2\ln x}{x^3}dx$$

$$= \int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2$$

$$Y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$= -\frac{1}{3}(\ln x)^3 x^2 + \frac{1}{2}(\ln x)(x^2\ln x)$$