Ch. 5 Series Solutions of 2nd order Linear Eqs.

5.2 Series Solution near an ordinary point, Part I

We consider the following diff. eq.

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

Defn. A point x_0 is called an ordinary point if

$$P(x_0) \neq 0$$

If $P(x_0) = 0$ then x_0 is called singular point.

If x_0 is an ordinary point, divide by $P(x)$,

$$y'' + p(x)\,y' + q(x)\,y = 0$$

where $p(x) = \dfrac{Q(x)}{P(x)}$, $q(x) = \dfrac{R(x)}{P(x)}$

are conts in an interval about x_0.

the IVP $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_0'$

has a unique solution.

In the neighbourhood of an ordinary point we look for a solution of the form

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Notice that $y(x_0) = y_0 = a_0$

$$y'(x_0) = y_0' = a_1$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-x_0)^n$$

Ex. Find series solution of $y''+y=0$ near $x_0=0$

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Subst. into the eq.

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + a_n \right] x^n = 0$$

We get the recurrence relation:

$$(n+1)(n+2)\, a_{n+2} + a_n = 0, \quad n=0,1,2,3,\dots$$

$$a_{n+2} = - \frac{a_n}{(n+1)(n+2)}.$$

a_0, a_1 are obtained from I.Cs

$$n=0: a_2 = -\frac{a_0}{2!}, \quad n=1 \Rightarrow a_3 = -\frac{a_1}{2.3} = -\frac{a_1}{3!}$$

$$n=2: a_4 = -\frac{a_2}{3.4} = \frac{a_0}{4!}, \quad n=3: a_5 = -\frac{a_1}{4.5} = \frac{a_1}{5!}$$

$$\vdots$$

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!}, \quad a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}$$

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= a_0 \cos x + a_1 \sin x.$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]$$

$$= a_0 y_1(x) + a_1 y_2(x).$$

Ex. Find a series solution of $y'' - xy = 0, \quad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2 \cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2 a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$$\Rightarrow \quad a_2 = 0$$

$$(n+1)(n+2) a_{n+2} - a_{n-1} = 0, \quad n = 1, 2, 3, \ldots$$

Rec. relation:

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)} \quad , \quad n = 1, 2, 3, \ldots$$

since $a_2 = 0 \Rightarrow a_5 = a_8 = a_{11} = \cdots = 0$

$n=1: a_3 = \frac{a_0}{2.3}$, $n=2: a_4 = \frac{a_1}{3.4}$

$n=3: a_5 = 0$, $n=4: a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6}$

$n=5: a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7}$, ---

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \frac{a_0}{180} x^6 + \cdots$$

$$= a_0 \left[1 + \frac{1}{6} x^3 + \frac{x^6}{180} + \cdots \right] + a_1 \left[x + \frac{1}{12} x^4 + \cdots \right]$$

$$y_1(x) = 1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots$$

$$y_2(x) = x + \frac{1}{12} x^4 + \cdots$$

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore y_1, y_2$ form a F.S. of solutions.

Ex Find series solution of $y''-xy=0$ near $x_0=1$.

$$y''-xy=0 \Leftrightarrow y''-(x-1+1)y=0$$

$$y''-(x-1)y-y=0$$

$$y=\sum_{n=0}^{\infty} a_n (x-1)^n, \quad y''=\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

subst. $$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} a_{n-1}(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1} - a_n\right](x-1)^n = 0$$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} - a_n = 0, \quad n=1,2,3,\dots$$

Recurrence Relation

$$a_{n+2} = \frac{a_{n-1} + a_n}{(n+1)(n+2)}, \quad n = 1, 2, 3, \cdots \quad \&, \quad a_2 = \frac{a_0}{2}$$

$$n=1 \Rightarrow a_3 = \frac{a_0 + a_1}{6} = \frac{a_0}{6} + \frac{a_1}{6}$$

$$n=2 \Rightarrow a_4 = \frac{a_1 + a_2}{12} = \frac{a_1}{12} + \frac{a_2}{12} = \frac{a_1}{12} + \frac{a_0}{24}$$

$$\vdots$$

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x-1)^3 + \left(\frac{a_1}{12} + \frac{a_0}{24}\right)(x-1)^4 + \cdots$$

$$= a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \cdots\right]$$

$$+ a_1\left[(x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots\right]$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

9. $(1+x^2)y'' - 4xy' + 6y = 0$, $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n , \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} , \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$+ \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4a_1 x - \sum_{n=2}^{\infty} 4n a_n x^n$$

$$+ 6a_0 + 6a_1 x + \sum_{n=2}^{\infty} 6 a_n x^n = 0$$

$$2a_2 + 6a_0 + (6a_3 - 4a_1 + 6a_1)x$$

$$+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n\right]x^n =$$

$$2a_2 + 6a_0 = 0 \Rightarrow a_2 = -3a_0$$

$$6a_3 + 2a_1 = 0 \Rightarrow a_3 = -\frac{1}{3}a_1$$

$$(n+1)(n+2)a_{n+2} + \underbrace{(n^2 - 5n + 6)}_{(n-2)(n-3)}a_n = 0$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n, \quad n = 2, 3, 4, \ldots$$

$$a_4 = 0, \quad a_5 = 0, \quad a_6 = 0, \; ---$$

$$y(x) = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3$$

$$= a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right)$$

$$y_1(x) = 1 - 3x^2, \quad y_2(x) = x - \frac{1}{3}x^3$$