5.3 Series Solution near Ordinary Point, Part II

Suppose that $y = \phi(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges for

$|x-x_0| < \rho$ is a series solution of

$$y'' + p(x)y' + q(x)y = 0$$

$\phi(x)$ has Taylor series $\sum_{n=0}^{\infty} \frac{\phi^{(n)}(x_0)}{n!} (x-x_0)^n$

It follows that $a_n = \frac{\phi^{(n)}(x_0)}{n!}$

we show how can we find the coefficients $a_0, a_1, \ldots$

directly from the differential equation.

$$a_2 = \frac{\phi''(x_0)}{2!} = \frac{1}{2!} y''(x_0) = \frac{1}{2!}\left(-p(x_0)y'(x_0) - q(x_0)y(x_0)\right)$$

$$= \frac{1}{2}\left(-p(x_0)a_1 - q(x_0)a_0\right).$$

differentiating the equation, we get

$$y''' + p'(x)y' + p(x)y'' + q'(x)y + q(x)y' = 0$$

$$\Rightarrow y'''(x_0) = -p'(x_0)y'(x_0) - p(x_0)y''(x_0) - q'(x_0)y(x_0) - q(x_0)y'(x_0)$$

$$a_3 = \frac{1}{3!} y'''(x_0).$$

12. Find the first four nonzero terms of each of two power series solutions of the diff-eq

$$(\cos x)y'' + xy' - 2y = 0, \quad \text{near } x_0 = 0$$

$x = 0 \Rightarrow \quad y''(0) + 0\,y'(0) - 2y(0) = 0.$

$$y''(0) = 2y(0) = 2a_0$$

$$a_2 = \frac{1}{2}y''(0) = \frac{1}{2}(2a_0) = a_0$$

diff. $(-\sin x)y'' + (\cos x)y''' + y' + xy'' - 2y' = 0$

$x = 0 \Rightarrow \quad 0y'' + y'''(0) + y'(0) + 0y'' - 2y'(0) = 0$

$$y'''(0) + a_1 - 2a_1 = 0$$

$$y'''(0) = a_1$$

$$a_3 = \frac{1}{3}y'''(0) = \frac{1}{3}a_1$$

$\vdots$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + a_0 x^2 + \frac{1}{3}a_1 x^3 + \cdots$$

$$= a_0\left[1 + x^2 + \cdots\right] + a_1\left[x + \frac{1}{3}x^3 + \cdots\right]$$

* Radius of Convergence of Series Solutions

We consider the diff eq. $y'' + p(x) y' + q(x) y = 0$

Defn. $p(x)$ and $q(x)$ are called analytic at x_0 if they have convergent power series

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n , \quad q(x) = \sum_{n=0}^{\infty} q_n (X - x_0)^n$$

in some open interval about x_0, and x_0 is ordinary point.

Thm. If x_0 is an ordinary point of the diff eq.

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

that is, if $p = Q/P$, $q = R/P$ are analytic at x_0, then the general solution of the above equation

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x), \quad a_0, a_1 \text{ const.}$$

The series solutions y_1 and y_2 form a F.S. of solutions.

The radius of convergence of the series of y_1 and y_2 is at least as large as the minimum of radius of convergence of series of p and q.

Rem. $y_1(x) = 1 + b_2(x-x_0) + \cdots$

$y_2(x) = (x-x_0) + c_2(x-x_0)^2 + \cdots$

$$W(y_1, y_2)(x_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

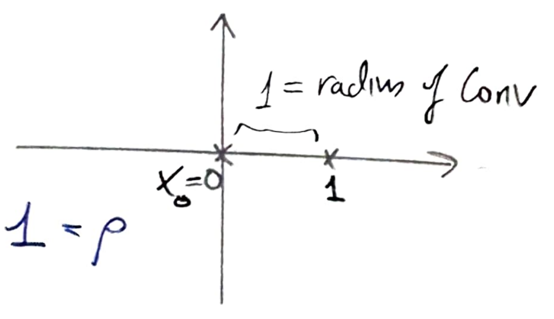
$\Rightarrow y_1, y_2$ is F.S. of Solutions.

Rem. The rational function of two polynomials $\frac{Q}{P}$ has Convergent Taylor Series about a point x_0 if $P(x_0) \neq 0$. and the radius of Convergence of the Taylor series equals the distance from x_0 to the nearest root, or zero, of P.

Ex $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$

this is a geometric Series.

$1 - x = 0 \Rightarrow x = 1$



distance from $x_0 = 0$ to $x = 1$ is $1 = \rho$

Ex. Find the radius of Convergence of Taylor Series of $\frac{1}{1+x^2}$ near $x_0=0$.

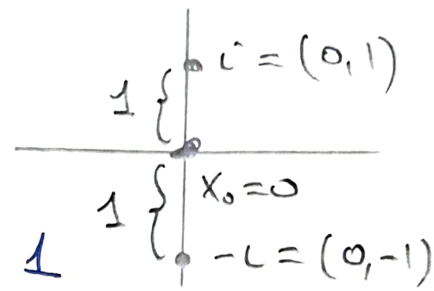
Method 1: Taylor series is a geometric Series

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$ Conv. if $|x^2|<$

$|x|<1 \Rightarrow \rho=1$

Method 2:

$1+x^2=0 \Leftrightarrow x=\pm i$



distance from $x_0=0$ to $\pm i$ is 1

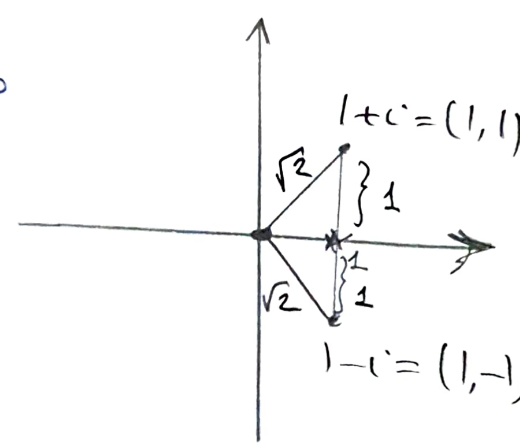
$\therefore$ radius of convergence $\rho=1$.

Ex. Find radius of Conv. of $\frac{1}{x^2-2x+2}$ about

a. $x_0=0$, b. $x_0=1$

$x^2-2x+2=0 \Leftrightarrow (x-1)^2+1=0$

$x=1\mp i$



a. $x_0=0$: $\rho=\sqrt{2}$

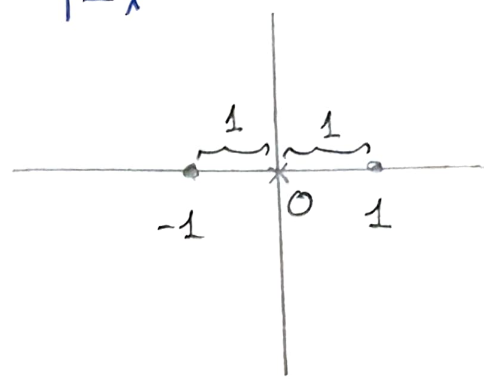
b. $x_0=1$: $\rho=1$

Ex Determine a lower bound for the radius of Conv. of series solutions about $x=0$ for

$$(1-x^2)\,y'' - 2xy' + 2y = 0$$

$$y'' - \frac{2x}{1-x^2}\,y' + \frac{2}{1-x^2}\,y = 0$$

$$p(x) = \frac{-2x}{1-x^2}$$



$$1-x^2=0 \Leftrightarrow x=\pm 1$$

$$q(x) = \frac{2}{1-x^2}, \quad 1-x^2=0 \Leftrightarrow x=\pm 1$$

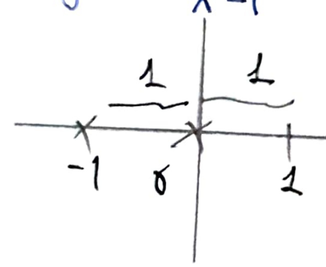
radius of Conv. for series of p, q is 1.

$\therefore$ radius of Conv. of series solution is $\rho \geq 1$.

Ex. Find lower bound of series Solution of $y'' + \frac{1}{x^2-1}y' + \frac{1}{x-\frac{1}{2}}y = 0$

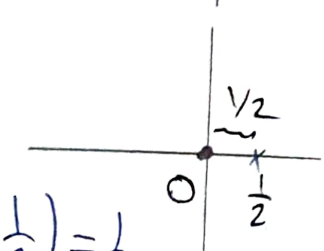
$$p(x) = \frac{1}{x^2-1}, \quad x^2-1=0 \Leftrightarrow x=\pm 1$$

$$\rho_1 = 1$$



$$q(x) = \frac{1}{x-\frac{1}{2}} : x-\frac{1}{2}=0 \Leftrightarrow x=\frac{1}{2}$$

$$\rho_2 = \frac{1}{2}$$



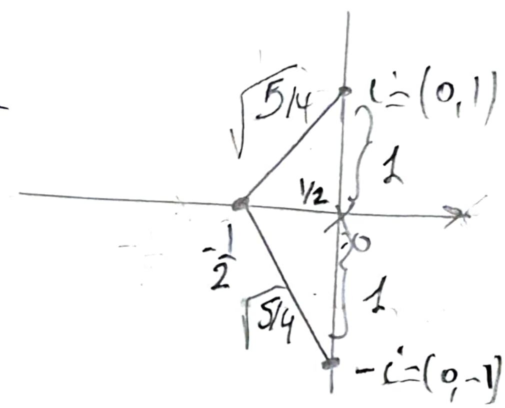
radius of Conv. of Solution $\rho \geq \min\left(1, \frac{1}{2}\right) = \frac{1}{2}$

$$\rho \geq \frac{1}{2}$$

Ex. Determine a lower bound for the radius of conv. of series solution of $(1+x^2)y'' + 2xy' + 4x^2 y = 0$ about (a) $x_0 = 0$, (b) $x_0 = -\frac{1}{2}$

$$y'' + \frac{2x}{1+x^2} y' + \frac{4x^2}{1+x^2} y = 0$$

$$p(x) = \frac{2x}{1+x^2}, \quad q(x) = \frac{4x^2}{1+x^2}$$



$$1+x^2 = 0 \iff x = \pm i$$

(a) for p: $\rho_1 = 1$, for q: $\rho_2 = 1$

$\therefore \rho \geq 1$

(b) for $x_0 = -\frac{1}{2}$: $\rho_1 = \frac{\sqrt{5}}{2}$, $\rho_2 = \frac{\sqrt{5}}{2}$

$\therefore \rho \geq \frac{\sqrt{5}}{2}$.

7- Find lower bound for radius of Conv. of Series soln of

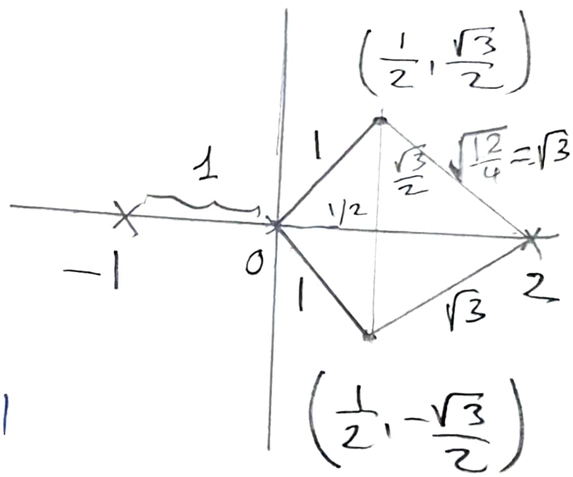
$$(1+x^3)\,y''+4xy'+y=0,\quad x_0=0,\quad x_0=2$$

$$y''+\frac{4x}{1+x^3}\,y'+\frac{1}{x^3+1}\,y=0$$

$$p(x)=\frac{4x}{1+x^3},\quad q(x)=\frac{1}{x^3+1}$$

$$x^3+1=(x+1)(x^2-x+1)=0$$

$$x=-1,\quad x=\frac{1\mp\sqrt{3}\,i}{2}$$



$x_0=0$: $\rho_1=\rho_2=1 \Rightarrow \rho\geq 1$

$x_0=2$: $\rho_1=\rho_2=\sqrt{3} \Rightarrow \rho\geq\sqrt{3}$

Rem. each time, we find the distance from x_0 to the roots of x^3+1 which are -1, $\frac{1\mp\sqrt{3}\,i}{2}$

ρ_1, ρ_2 equals the distance from x_0 to the nearest root. and $\rho\geq\min(\rho_1,\rho_2)$.