# 5.4 Euler Equations

Euler equations are diff eqs of the form:

$$x^2 y'' + \alpha x y' + \beta y = 0 \,, \quad x > 0 .$$

let $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$

$$x^2 \left( r(r-1) x^{r-2} \right) + \alpha x \left( r x^{r-1} \right) + \beta x^r = 0$$

$$r(r-1) x^r + \alpha r x^r + \beta x^r = 0$$

$$\left( r(r-1) + \alpha r + \beta \right) x^r = 0$$

$\therefore$ $y = x^r$ is a solution if and only if

$r$ is a solution of the equation

$$r(r-1) + \alpha r + \beta = 0$$

$$r^2 + (\alpha - 1) r + \beta = 0$$

$$r = \frac{-(\alpha-1) \mp \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

If $r_1, r_2$ are real, distinct $r_1 \neq r_2$, then

$$y_1(x) = x^{r_1}, \quad y_2(x) = x^{r_2}$$

the general solution is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

Ex. Find the general solution of $2x^2y'' + 3xy' - y = 0$

$$2r(r-1) + 3r - 1 = 0$$

$$2r^2 + r - 1 = 0 \Leftrightarrow (2r-1)(r+1) = 0$$

$$r_1 = \frac{1}{2}, \quad r_2 = -1$$

$$y(x) = c_1 x^{1/2} + c_2 x^{-1}, \quad x > 0.$$

Ex. $x^2y'' - 2xy' + 2y = 0, \quad x > 0$

$$r(r-1) - 2r + 2 = 0 \Leftrightarrow r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0 \Rightarrow r_1 = 1, \ r_2 = 2$$

$$y(x) = c_1 x + c_2 x^2.$$

Case2 $r_1 = r_2 = r \Rightarrow y = x^r$ is one solution of

$$x^2y'' + \alpha x y' + \beta y = 0. \quad , \quad y_1(x) = x^{-\frac{(\alpha-1)}{2}}$$

We use reduction of order to find $y_2(x)$.

$$y'' + \alpha \frac{x}{x^2} y' + \frac{\beta}{x^2} y = 0$$

$$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0$$

$$e^{-\int \frac{\alpha}{x} dx} = e^{-\alpha \ln x} = \frac{1}{x^\alpha}$$

$$V(x) = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$$

$$= \int \frac{1}{x^{1-\alpha}} \frac{1}{x^\alpha} dx = \int \frac{dx}{x} = \ln x$$

$$y_2(x) = y_1(x) \, v(x)$$

$$= x \ln x.$$

Ex. Solve $x^2y'' + 5xy' + 4y = 0, \quad x > 0.$

$$r(r-1) + 5r + 4 = 0$$

$$r^2 + 4r + 4 = 0 \iff (r+2)^2 = 0$$

$$r = -2$$

$$y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x.$$

Ex. $x^2y'' - xy' + y = 0, \quad x > 0, \quad y(1) = 1,$
$y'(1) = 2$

$$r(r-1) - r + 1 = 0$$

$$r^2 - 2r + 1 = 0 \iff (r-1)^2 = 0 \Rightarrow r = 1$$

$$y = c_1 x + c_2 x \ln x.$$

$$y(1) = c_1 = 1$$

$$y'(x) = c_1 + c_2 \ln x + c_2$$

$$y'(1) = c_1 + c_2 = 2 \Rightarrow c_2 = 1$$

$$y(x) = x + x \ln x, \quad x > 0.$$

Case 3. Complex roots : $r = \lambda \mp i\mu$

$$x^{\lambda + i\mu} = x^{\lambda} x^{i\mu} = x^{\lambda} e^{\ln x^{i\mu}}$$

$$= x^{\lambda} e^{i\mu(\ln x)}$$

$$= x^{\lambda} \cos(\mu \ln x) + i x^{\lambda} \sin(\mu \ln x)$$

$\therefore$ Real Solutions :

$$y_1(x) = x^{\lambda} \cos(\mu \ln x) , \quad y_2(x) = x^{\lambda} \sin(\mu \ln x).$$

general Solution :

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x).$$

Ex. Solve $x^2 y'' + x y' + y = 0$

$$r(r-1) + r + 1 = 0$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i, \quad \begin{matrix} \lambda = 0 \\ \mu = 1 \end{matrix}$$

$$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

Remark: $x^2y'' + \alpha x y' + \beta y = 0, \quad x < 0$

(1) $y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}$

(2) $y(x) = c_1 |x|^{r} + c_2 |x|^{r} \ln|x|$

(3) $y(x) = c_1 |x|^{\lambda} \cos(\mu \ln|x|) + c_2 |x|^{\lambda} \sin(\mu \ln|x|)$.

Exercise,

(4) $x^2y'' + 3xy' + 5y = 0$.

$$r^2 + 2r + 5 = 0 \Leftrightarrow r^2 + 2r + 1 + 4 = 0$$

$$(r+1)^2 + 4 = 0 \Leftrightarrow r = -1 \mp 2i \qquad \begin{matrix} \lambda = -1 \\ \mu = 2 \end{matrix}$$

$$y(x) = c_1 x^{-1} \cos(2\ln x) + c_2 x^{-1} \sin(2\ln x).$$

(6) $(x-1)^2 y'' + 8(x-1)y' + 12y = 0$

$$r^2 + 7r + 12 = 0$$

$$(r+4)(r+3) = 0 \Rightarrow r = -3, -4$$

$$y(x) = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}.$$

9. $x^2y'' - 5xy' + 9y = 0$

$r^2 - 6r + 9 = 0 \Leftrightarrow (r-3)^2 = 0, \Rightarrow r = 3$

$y(x) = C_1 x^3 + C_2 x^3 \ln x.$

37. Find $\gamma$ so that the solution of the IVP

$x^2y'' - 2y = 0, \quad y(1) = 1, \ y'(1) = \gamma$

is bounded as $x \to 0$

$r^2 - r - 2 = 0 \Leftrightarrow (r-2)(r+1) = 0,$

$r = -1, r = 2$

$y(x) = C_1 x^{-1} + C_2 x^2, \quad y'(x) = -C_1 x^{-2} + 2C_2 x$

$y(x)$ will be bounded as $x \to 0$ if $C_1 = 0$

$y(1) = C_1 + C_2 = 1$

$y'(1) = -C_1 + 2C_2 = \gamma$

$\Rightarrow 3C_2 = \gamma + 1$

$C_2 = \frac{\gamma + 1}{3}$

if $C_1 = 0 \Rightarrow C_2 = 1 = \frac{\gamma}{2} \Leftrightarrow \boxed{\gamma = 2}$

$\therefore y(x)$ is bounded as $x \to 0$ if $\gamma = 2.$