Regular Singular Points

Consider the differential Equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Suppose that x_0 is a singular point; that is

$P(x_0) = 0$, and at least one of Q and R

is not zero at x_0.

Defn. A singular point x_0 is called a regular singular point if

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{is finite}$$

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \quad \text{is finite}$$

otherwise, x_0 is called an irregular singular point.

Ex. $(1-x^2)y'' - 2xy' + 2y = 0$

$P(x) = 1-x^2$, $Q(x) = -2x$, $R(x) = 2$.

$P(x) = 1-x^2 = 0 \Rightarrow x = \pm 1$ are singular points

$x=1$: $\displaystyle \lim_{x\to 1} \frac{(x-1)\,Q(x)}{P(x)} = \lim_{x\to 1} \frac{(x-1)(-2x)}{1-x^2}$

$\displaystyle = \lim_{x\to 1} \frac{(x-1)(-2x)}{(1-x)(1+x)} = \lim_{x\to 1} \frac{2x}{1+x} = 1$

$\displaystyle \lim_{x\to 1} (x-1)^2 \frac{R(x)}{P(x)} = \lim_{x\to 1} (x-1)^2 \frac{2}{(1-x)(1+x)}$

$\displaystyle = \lim_{x\to 1} \frac{-2(x-1)}{1+x}$

$= 0$

$\therefore$ $x=1$ is a regular singular point

$x=-1$: $\lim\limits_{x\to -1} (x+1)\dfrac{Q(x)}{P(x)} = \lim\limits_{x\to -1} \dfrac{(x+1)(-2x)}{(1-x)(1+x)}$

$= \lim\limits_{x\to -1} \dfrac{-2x}{1-x} = 1$

$\lim\limits_{x\to -1} (x+1)^2 \dfrac{R(x)}{P(x)} = \lim\limits_{x\to -1} \dfrac{2(x+1)^2}{(1-x)(1+x)} = 0$

$x=-1$ is regular singular point.

Ex. $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

$P(x) = 2x(x-2)^2 = 0 \Rightarrow x = 0, 2$ singular points

$x=0$: $\lim\limits_{x\to 0} x \cdot \dfrac{3x}{2x(x-2)^2} = 0$

$\lim\limits_{x\to 0} x^2 \dfrac{(x-2)}{2x(x-2)^2} = 0$

$\Rightarrow$ $x=0$ is regular singular point

$x=2$ $\lim\limits_{x\to 2} (x-2)\dfrac{3x}{2x(x-2)^2} = \lim\limits_{x\to 2} \dfrac{3}{2x-2}$ DNE

$x=2$ is irregular singular point.

5.5 Series Solution near Regular Singular Point.

Assume that $x=0$ is a regular singular point of the diff eq.

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$\therefore \lim_{x \to 0} x\frac{Q(x)}{P(x)}$, $\lim_{x \to 0} x^2\frac{R(x)}{P(x)}$ are finite

A series solution of the diff eq. has the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

Ex. Solve $2x^2y'' - xy' + (1+x)y = 0$

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0}\frac{-x^2}{2x^2} = -\frac{1}{2}$$

$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0}\frac{x^2(1+x)}{2x^2} = \frac{1}{2}$$

$\therefore x=0$ is a regular singular point

$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}y = 0$ is called the Euler

equation corresponding to $2x^2y'' - xy' + (1+x)y = 0$

$$2x^2y'' - xy' + y = 0$$

$$2r(r-1) - r + 1 = 0$$

$2r^2 - 3r + 1 = 0$ is called indicial equation

$$(2r-1)(r-1) = 0$$

$r_1 = 1$, $r_2 = \frac{1}{2}$ are called the exponents

at singularity.

let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Substitute into the diff. eq.

$$\sum_{n=0}^{\infty} 2 a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{\sum_{n=1}^{\infty} a_{n-1} x^{n+r}} = 0$$

$$a_0 \left(2r(r-1) - r + 1\right) x^r +$$

$$\sum_{n=1}^{\infty} \left[\left[2 (n+r)(n+r-1) - (n+r) + 1 \right] a_n + a_{n-1} \right] x^{n+r} = 0$$

$\Rightarrow$ $2r(r-1) - r + 1 = 0 \Leftrightarrow 2r^2 - 3r + 1 = 0$ indicial eq.

$(2r-1)(r-1) = 0 \Rightarrow r_1 = 1, \; r_2 = \frac{1}{2}$ exponents at singularity.

$$\left[2(n+r)(n+r-1)-(n+r)+1\right]a_n + a_{n-1} = 0$$

$$\left[2(n+r)^2-3(n+r)+1\right]a_n + a_{n-1} = 0 \quad , \quad n=1,2,3,\ldots$$

$$a_n = \frac{-a_{n-1}}{\left(2(n+r)-1\right)\left(n+r-1\right)} \quad , \quad n \geq 1$$

$$r_1 = 1 : \quad a_n = \frac{-a_{n-1}}{\left(2(n+1)-1\right)\left(n+1-1\right)} \quad , \quad n \geq 1$$

$$a_n = -\frac{a_{n-1}}{n(2n+1)} \quad , \quad n \geq 1$$

$$a_1 = -\frac{a_0}{1.3} \quad , \quad a_2 = -\frac{a_1}{2.5} = \frac{a_0}{3.5.1.2}$$

$$a_3 = -\frac{a_2}{3.7} = \frac{a_0}{1.2.3.3.5.7}$$

$$\vdots$$

$$a_n = \frac{(-1)^n}{n!\left[3.5.7\ldots(2n+1)\right]}$$

$$a_n = \frac{(-1)^n}{n![3.5.7\cdots(2n+1)]} \cdot \frac{2.4.6.\cdots 2n}{2.4.6.\cdots 2n}$$

but $2.4.6\cdots 2n = 2^n n!$

$$a_n = \frac{(-1)^n \, 2^n n!}{n!(2n+1)!} = \frac{(-1)^n 2^n}{(2n+1)!}$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^{n+1}$$

$$r_2 = \frac{1}{2} : a_n = -\frac{a_{n-1}}{2n(n-\frac{1}{2})} = -\frac{a_{n-1}}{n(2n-1)}, n \geq 1$$

$$a_1 = -\frac{a_0}{1.1}, \quad a_2 = -\frac{a_1}{2.3} = +\frac{a_0}{(1.2)(1.3)}$$

$$a_3 = \frac{-a_2}{3.5} = -\frac{a_0}{(1.2.3)(1.3.5)}$$

$$a_n = \frac{(-1)^n a_0}{n![1.3.5\cdots(2n-1)]}$$

$$a_n = \frac{(-1)^n}{n!\,[1.3.5\cdots(2n-1)]} a_0 \times \frac{2.4.6\cdots 2n}{2.4.6\cdots 2n}$$

$$= \frac{(-1)^n\, 2^n\, n!}{n!\,(2n)!}$$

$$a_n = \frac{(-1)^n\, 2^n}{(2n)!}$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(2n)!} x^{n+\frac{1}{2}}$$

The general solution is:

$$y(x) = C_1 \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(2n+1)!} x^{n+1} + C_2 \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(2n)!} x^{n+\frac{1}{2}}$$