1.2 Solutions of Some Differential Eqs.

Remember that the free fall and owl/mice differential equations

$$v' = \frac{\partial v}{\partial t} = 9.8 - 0.2v \quad \text{and}$$

$$p' = \frac{\partial p}{dt} = 0.5p - 450$$

They are of the form $\boxed{y' = ay - b}$

To solve $\boxed{p' = 0.5p - 450}$ (*) , we use methods of calculus as follows:

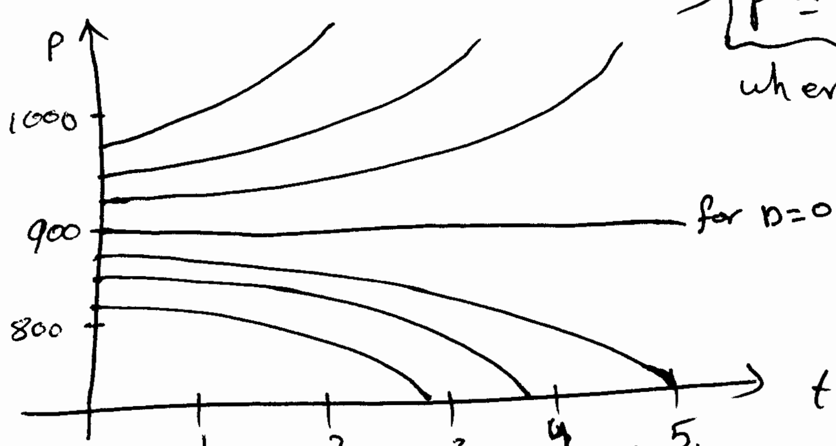
the differential equation

$$\frac{\partial p}{\partial t} = 0.5(p - 900) \Rightarrow \int \frac{\frac{dp}{dt}}{p-900} = \int 0.5 \Rightarrow$$

$$\ln|p-900| = 0.5t + c \Rightarrow |p-900| = e^{0.5t+c} \Rightarrow$$

$$p - 900 = \pm e^{0.5t+c} \Rightarrow \boxed{p = 900 + D e^{0.5t}}$$

where $D = \pm e^{c}$ constant



Integral curves (Graph solution)

For several values of the constant D

i.e infinitly many solution for (*) corresponding to the arbitrary constant D

* Initial Conditions:-

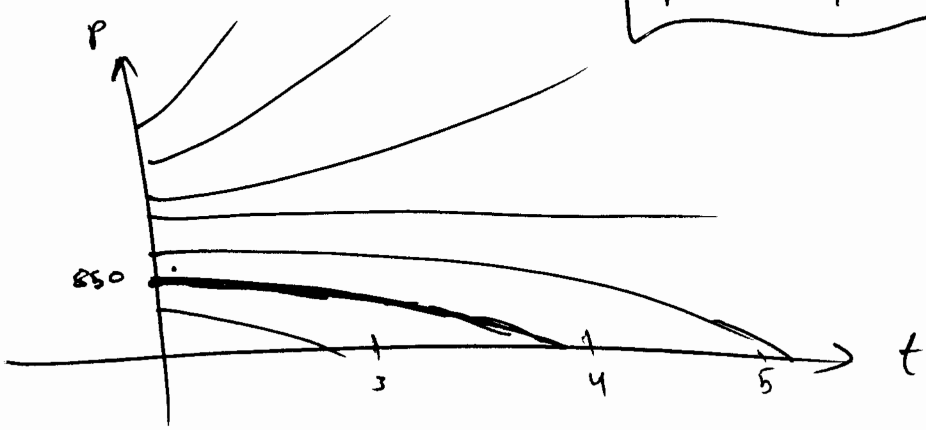
If we know a point that lies on the graph of the solution, then we determine a unique solution. Such point is called the initial condition.

→ Recall the solution $\boxed{p(t) = 900 + D e^{0.5t}}$ *

→ Suppose that the mice population starts at 850 i.e $p(0) = 850$, Then from * we have

$$850 = 900 + D e^{0.5(0)} \Rightarrow \boxed{D = -50}$$

Hence, the solution becomes $\boxed{p(t) = 900 - 50 e^{0.5t}}$



Note that the differential equation
$p' = 0.5p - 450$ together with the initial condition $p(0) = 850$ form an initial value problem whose solution is $p(t) = 900 - 50 e^{0.5t}$.

* Initial Value Problem (IVP):

Solve the following initial value problem

$$y' = ay - b \;,\; y(0) = y_0$$

We use the methods of calculus as follows

$$\frac{dy}{dt} = a\left[y - \frac{b}{a}\right] \Rightarrow \int \frac{\frac{dy}{dt}}{y - \frac{b}{a}} = \int a$$

$$\Rightarrow \ln\left|y - \frac{b}{a}\right| = at + c \Rightarrow \left|y - \frac{b}{a}\right| = e^{at+c}$$

$$\Rightarrow y - \frac{b}{a} = \pm e^{at+c} \Rightarrow \boxed{y = \frac{b}{a} + D e^{at}}$$

where the constant D is equal to $\pm e^{c}$

* Using the initial condition $y(0) = y_0$ $\Rightarrow$

$$y_0 = \frac{b}{a} + D e^{a(0)} \Rightarrow y_0 = \frac{b}{a} + D$$

$$\Rightarrow \boxed{D = y_0 - \frac{b}{a}}$$

Hence,

$$\boxed{y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}}$$

this is called general solution: it contains all possible solutions

* Recall the Initial Value problem

$$y' = ay - b \quad , \quad y(0) = y_0$$

* Remember that the equilibrium solution (when $y' = 0$) is $0 = ay - b \Rightarrow y(t) = \frac{b}{a}$

* Remember that the general solution of the IVP above is

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

Note the following solution behavior:

* If $y_0 = \frac{b}{a}$ then $y(t) = \frac{b}{a}$ "constant"

* If $y_0 > \frac{b}{a}$ and $a > 0$, then $y(t)$ increases exponentially without bound

* If $y_0 > \frac{b}{a}$ and $a < 0$, then $y(t)$ decreases (decays) exponentially to $\frac{b}{a}$

* If $y_0 < \frac{b}{a}$ and $a > 0$, then $y(t)$ decreases exponentially without bound.

* If $y_0 < \frac{b}{a}$ and $a < 0$, then $y(t)$ increases asymptotically to $\frac{b}{a}$

H.W page 14 : 1, 3, 7, 12, 13, 14, 15, 17 go to the end of page 11

Example Consider a falling object of mass = 10 kg and drag coefficient $\gamma = 2$ kg/sec.

$$\frac{\partial v}{\partial t} = 9.8 - 0.2v$$

Suppose this object is dropped from a height of 300 m.

(a) Find its velocity at any time t.

(b) How long until it hits ground and how fast will it be moving then?

Solution: The IVP is $v' = 9.8 - 0.2v$, $v(0) = 0$

(a) Using the general solution

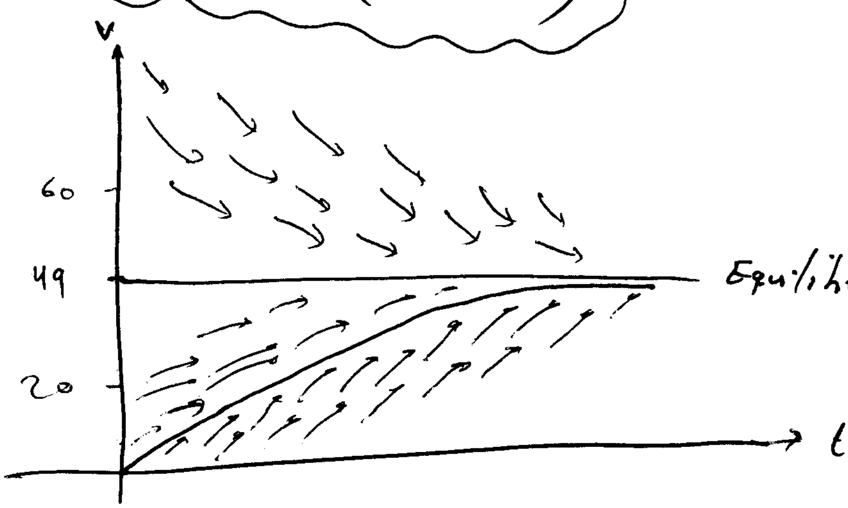
$$v(t) = \frac{b}{a} + \left[v_0 - \frac{b}{a}\right]e^{at} \quad \text{for } v' = av - b$$

$a = -0.2$

$b = -9.8$

$$v(t) = \frac{9.8}{0.2} + \left[0 - \frac{9.8}{0.2}\right]e^{-0.2t}$$

$$v(t) = 49\left(1 - e^{-0.2t}\right)$$

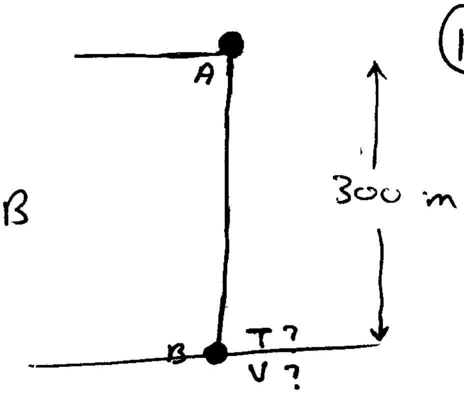


Equilibrium Solution

$\frac{b}{a} = 49$

$V_0 = 0 < 49$ and $a = -0.2 < 0$ then $v(t)$ increases asymptotically to 49.

(b) We need to find the time at position B and the velocity at position B



→ let $x(t)$ be the distance that the object has fallen at time t

$x(0) = 0$ and $x(T) = 300$

this the time when the object touch the grown.

$x'(t) = V(t) = 49\left(1 - e^{-0.2t}\right)$ since $V = \frac{dx}{dt}$ at moment t

$x(t) = 49t + 49 \times \frac{1}{0.2} e^{-0.2t} + c$

$$\boxed{x(t) = 49t + 245\, e^{-0.2t} + c}$$

$x(0) = 0 \Rightarrow 0 = 0 + 245 + c \Rightarrow \boxed{c = -245}$

$$\Rightarrow x(t) = 49t + 245\, e^{-0.2t} - 245$$ ⇒ This is the distance at any time t

$x(T) = 300 = 49T + 245\, e^{-0.2T} - 245$

$T \approx 10.51$ sec This is the time when the object touch the ground

Hence, the velocity of the object at $T = 10.51$ is

$V(10.51) = 49\left(1 - e^{-0.2 \times 10.51}\right) \approx 43.01$ m/sec.