1.3 Classification of Differential Equations

The main purpose of this course is to

- discusss properties of solutions of Differential equations
- present methods of finding solutions or approximating solutions

To provide a framework for our dissicussion, we describe several ways to classify differential Equations;

(1) * Ordinary Differential Equations (ODE)

- The unknown function depends on a single independent variable
  - $\Rightarrow$ only ordinary derivatives appear in the equation
  - $\Rightarrow$ In this case, the equation is called ODE

Examples $\quad \dfrac{dv}{dt} = 9.8 - 0.2v \qquad \dfrac{dp}{dt} = 0.5p - 450$

(2) * Partial Differential Equations (PDE)

- The unknown function depends on several independent variables
  - $\Rightarrow$ partial derivatives appear in the equation
  - $\Rightarrow$ In this case, the equation is called PDE

Examples $\quad \alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ "heat equation"

constants — are linear PDE

$a^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2 u(x,t)}{\partial t^2}$ "wave equation"

(3) * Systems of Differential Equations

- If the Differential Equation has a single unknown function, then one equation is sufficient.
- If there are two or more unknown functions that need to be found, then a system of equation is required.

Example: Lotka-Volterra (predator-prey) equations

non linear ODE

$$\frac{dx}{dt} = ax - \alpha xy$$

$$\frac{dy}{dt} = -cy + \gamma xy$$

a, α, c, γ are constants that depend on a particular species we study.

where $x(t)$ and $y(t)$ are the perspective populations of the prey and predator species.

Order of Differential Equations

The order of a differential Equation is the order of the highest derivative that appears in the equation.

Examples:

$y' - y = 0$: 1^{st} order ODE

$2y'' - 5y' + 3t = 0$: 2^{nd} order ODE

$\frac{d^6y}{dt^6} + \frac{d^3y}{dt^3} - 5 = e^{\sqrt{2}t}$: ODE of order 6

$u_{xx} - u_{yy} - \cos t = 0$: PDE of order 2

$\Rightarrow$ We assume the possibility to solve a given ODE for the highest derivative, i.e.

$$y^{(n)}(t) = f\left(t, y, y', y'', \dots, y^{(n-1)}\right)$$

$y = y(t)$
$y' = y'(t)$
$\vdots$
$y^n = y^n(t)$

To simplify notations

Examples :

$$y' = y$$

$$y'' = \frac{1}{2}\left(5y' - 3t\right)$$

$$\frac{\partial^6 y}{d t^6} = -\frac{d^3 y}{d t^3} + e^{\sqrt{2}\,t} + 5 \quad \dots$$

Linear and Nonlinear Differential Equations

In general, we write the DE of order n in the form

$$F\left(t, y, y', y'', \dots, y^{(n)}\right) = 0$$

$\rightarrow$ F is called linear if it is linear in the variables $y, y', y'', \dots, y^{(n)}$

Note that F can be ODE or PDE

The general linear ODE of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_n(t)\, y = g(t)$$

Example

$u_{xx} - u\,u_{yy} - \sin t = 0$ — nonlineare PDE

$y'' - 5y' + t^2 = 0$ — linear ODE

$\frac{d^2 y}{d t^2} - e^{y} \frac{dy}{dt} = 5t$ — non linear ODE.

$x y' + 2y = \sin x$ — 1^{st} order linear

Solutions to Differential Equations

* A solution $\phi(t)$ to the following ODE

$$y^{(n)}(t) = f(t, y, y', y'', \ldots, y^{(n-1)})$$

on the interval $\alpha < t < \beta$ is a function such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)})$$

Example: Verify the following solutions of the ODE

$$y'' + y = 0$$

① $y_1(t) = \sin t$ ② $y_2(t) = -\cos t$ ③ $y_3(t) = 2\sin t$

Three important questions in the study of DEs:

1) Is there a solution? (Existence)
2) If there is a solution, is it unique? (Uniqueness)
3) If there is a solution, how do we find it?
(Analytical Solution, Numerical Approximation)

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